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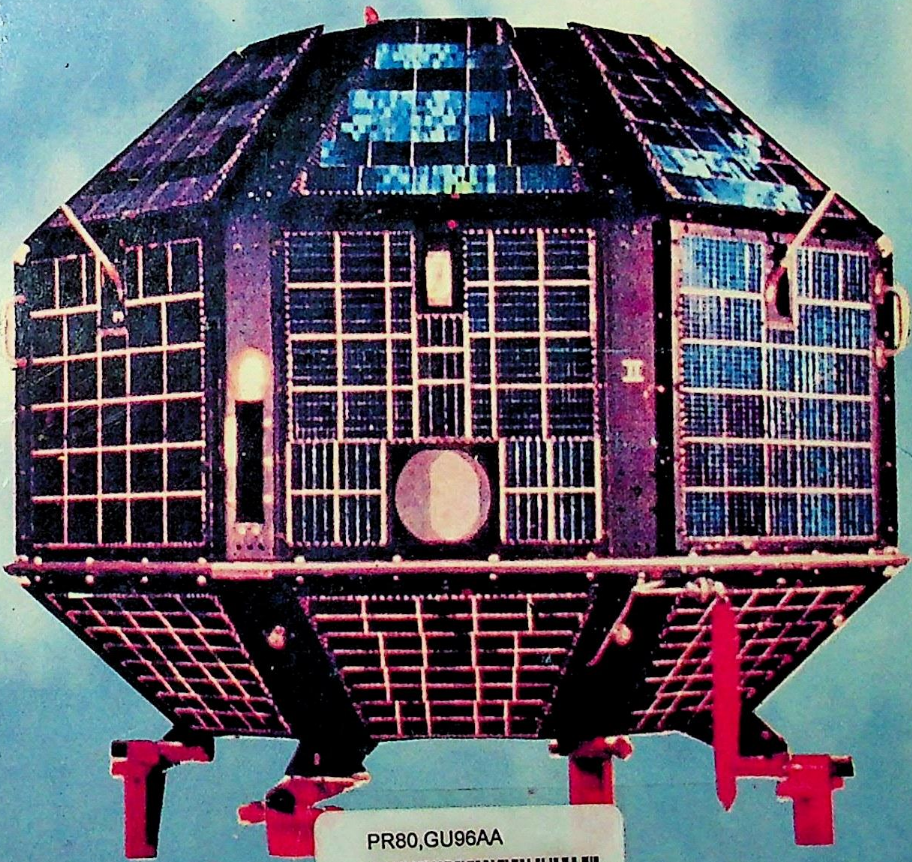
गुरुकुल कांगड़ी विज्ञान पत्रिका

Gurukula Kangri Vijñāna Patrikā

आर्यभट

ĀRYABHAṬA



गुरुकुल कांगड़ी विश्वविद्यालय, हरिद्व

Gurukula Kangri Vishwavidyalaya, Haridv

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट
Gurukula Kangri Vijñāna Patrikā Āryabhata

संदेश

कुलपति (डॉ०) धर्मपाल महोदय

गुरुकुल कांगड़ी विश्वविद्यालय ने स्थापना काल से ही वैज्ञानिक विषयों के अध्ययन व उपाधि निरपेक्ष अनुसंधान की दिशा में उल्लेखनीय कार्य किया है; विज्ञान के प्रारम्भिक अध्येताओं के लिए हिन्दी में पुस्तकों का लेखन व प्रकाशन तब सम्पन्न कराया जब हिन्दी में वैज्ञानिक विषयों की पारिभाषिक शब्दावली भी उपलब्ध नहीं थी। स्वामी श्रद्धानन्द जी और उनके शिष्य पत्रकारों ने साहित्यक व राजनीतिक पत्रकारिता के साथ-साथ विज्ञान पत्रकारिता में भी रूचि ली। आर्य समाज के साथ प्रत्यक्ष और परोक्ष से जुड़े हुए विभिन्न विश्वविद्यालयों के और विश्वविद्यालय से बाहर के जिन वैज्ञानिकों का ऐतिहासिक महत्व है उसमें स्वामी (डॉ०) सत्य प्रकाश जी, फूलदेव सहाय वर्मा तथा रामदास गौड़ का नाम उल्लेखनीय है।

गुरुकुल को विश्वविद्यालय का दर्जा मिल जाने के बाद यहां के विद्वानों ने यह अनुभव किया कि प्राचीन भारतीय विज्ञान के साथ-साथ आधुनिक विज्ञान की उपलब्धियों का हिन्दी में विवेचन व विश्लेषण प्रस्तुत किया जाय। शोध स्तरीय लेखन के साथ-साथ विज्ञान को लोकप्रिय बनाने के लिये हिन्दी में विज्ञान विषयक पत्रिकाओं का प्रकाशन किया जाय। यह कार्य हिन्दी और अंग्रेजी के माध्यम से विज्ञान महाविद्यालय के प्राध्यापकों ने बड़ी निष्ठा के साथ किया। विज्ञान के राष्ट्रीय व अंतर्राष्ट्रीय ख्याति प्राप्त विद्वानों ने इन पत्रिकाओं की प्रशंसा भी की। सम्प्रति *प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका* डॉ० एस० एल० सिंह के संपादन में सफलता पूर्वक नियमित रूप से विगत एक दशक से प्रकाशित हो रही है।

मुझे जानकर प्रसन्नता हो रही है कि अब विज्ञान महाविद्यालय से आर्यभट नाम की पत्रिका प्रकाशित होने जा रही है। यह पत्रिका भारतीय विज्ञान तथा आधुनिक विज्ञान की उपलब्धियों के तुलनात्मक अध्ययन की दिशा में एक सशक्त माध्यम का कार्य करेगी तथा विज्ञान की दुरुह ग्रंथियों को सरल से सरल रूप में प्रस्तुत करने का प्रयत्न करेगी। मुझे विश्वास है कि भारत और भारत से बाहर के इस विषय में रूचि लेने वाले विद्वानों का योगदान इस पत्रिका को मिलेगा। यह एक सार्वभौम लोक कल्याणकारी अभियान का रचनात्मक कार्य होगा। इन शब्दों के साथ मैं पत्रिका के प्रवेशांक पर सम्पादक मण्डल व लेखक मंडल को हार्दिक बधाई देता हूँ।

धर्ममाल

दिनांक १० फरवरी १९९८

कुलपति

गुरुकुल कांगड़ी विश्वविद्यालय, हरिद्वार

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट
Gurukula Kangri Vijñāna Patrika Aryabhata

विषय सूची

CONTENTS

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1998 Vol 1(2)

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HINDU SAṀKHYA PADDHATI

SAILESH DASH GUPTA*

The word *saṁkhyā* appears in *upanisada*, The same is used there to explain philosophical concept. The normal imports of *saṁkhyā,* are Teckoning, summing up, numeration, calculation, a numeral and number. *Saṁkhyāna* means enumeration, counting numbering. *Saṁkhyāṅk* denotes cipher (not zero). According to *Buddhists Saṁakhyā* refers to a large number. *Jainas* designate science of numbers by *saṁkhyāna*. A spirit of devotion and veneration is attached to the word. *Mahōvirācārya* designates Lord Mahavira as *saṁkhyā-Jñāna-Pradipena* as highest mark of distinction. *Saṁkhāyana* occurs in *saṁkhyāna* Āraṇyoka where *Guṇākhya* is given the authority for the work. *Gṛhya-sūtras* calls a teacher as *suyajña saṁkhyāyana*.

Later philosophers and religio-Philosopher Buddha attempted to reach to the essence of *saṁkhyā* and its associate *sūnya*. One of the six traditional system of Philosophy is Kapila's *Sāṁakhya*. Among the twenty-five categories for enquiring reality, one method relies on the model of numeration or *saṁkhyā*. The *Sāṁkhya* philosophy aspires for ultimate knowledge. The word *Sāṁkhya* is derived from *Sāṁkhya*. The *Advaita* philosophy centres around *Bṛahma* as one and only one and ignores any other thing. According to them *śūnya* is *svarūpahinam* (that which never in existence) and also *nirūpākhyam* (which is never assertible).

Mahāyana Budhism is recognised as nihilism. it does not believe in the position of *Bṛahma*. Instead, if places *Śūnya* in its place being the source and dissolution of every thing. Its followers are recognised as teachers of *Śūnya-Vādins*. *Śūnyavāda* of Buddhism spells ultimate truth.

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They used *Śūnyam* and *Śūnyata* to explain *'Yathābhūtam'* for indication and incitation, state of awareness.

In this scenerio the Tantric Philosophy is more direct. It distinguishes between a small circle representing *Śūnya* and a *bindu*. A circle demonstrates a completeness and separateness. When a circle is drawn, if distinctly divides the space in two regions, inside and outside. Again it can be conceived as a continuous line, without any assignable point of origin. Moreover, its symmetrical nature appeals the tantric people and they actually use the form for their worship. In the final stage they consider *bindu* in every *yantra* (meditational device) and place it at the centre. To them *bindu* is the highest concentrated energy and this will be revealed only to the sādhaka. Who after crossing various hurdles reaches his goal at the *bindu*. This *bindu* is the first principle and 'beyond which energy cannot be condensed'. The total conception of *bindu* is 'Purna' i.e. full. There is no negative content in the *bindu*. Thus this extreme philosophical or religious altitude towards *bindu* creates the full concept of *śūnya* in mathamatics and philosophy. It is also known, previously *śūnya* was represented by a dot, the physical existence was considered more important and vital for understanding and use it in various computations. Moreover *śūnya* having equal weight as other numerals and as such the shape and size are justified under all considerations. Thus many theories and knowledge shaped Hindu *śūnya*. After philosophizing on different aspects of *śūnya* and *saṁkhyā* it reminds us the off quoted lines of *Upaniṣada*.

'Tamaso mā jyotirgamaya'.

The Hindu *śūnya* forms the centre of dispute as to its origin. The problem arises due to the fact that *Vedic* terms do not include the word *'śūnya'* - the word name presently used. Now the concept of *bindu* and representation is observed in all *yantras* of the Tantric school. Many equivalent words are suggested for *śūnya* as empty, devoid, nothing,

non-existent *void*. The words imply negation. But *śūnya* is not associated with any such terms. *Śūnya* exists in a positive sense. If any body prefers to call *śūnya* by such terms, he will make the same mistake as Euclid, defining a point as dimensionless. *śūnya* is a number like 1 to 9 with special characteristics as $a \times o = 0$ and $a^0 = b^0 = c^0 = 1$.

We observe *śūnya* reigning in full power in *Āryabhaṭa's Āryabhaṭiya*. The Bakhshali manuscript has exhibited eight zeros in a folio. *Yovanajātaka* use decimal place value system with *śūnya*. *Pingala Chandaḥ - śūtra* shows : Place two when halved, and unity is subtracted then place zero. This shows $2 \div 2 = 1$, $1-1 = 0$. This *śūnya* is obtained by subtracting one from one and so it is the genuine *śūnya* or *zero*. Bhadrabahu used a word *'thibuqa'* for *śūnya*, according to *Hemchandra*. We will soon observe *Āryabhaṭa* using *'kha'* (sky) for zero. It is claimed that the concept of *śūnya* first emanated from grammarian Paṇini with the full connotation of the word. In Vedic Mathematics the author interprets the use of *rṣa* in *Atharva Veda* for zero. Moreover it is observed that words like *kham*, Vyoma, *pūrna*, having similar meaning, derived from the *vedas* were used to denote *śūnya*. Thus we get a continuous and systematic use of zero, with various name in course of time.

The whole mathematical science of the Hindu is built with only one system of numerals. No-where there is mention of any other notation. The system prevailed from the Vedic age to the present day. Al-khwarizmi translated Hindu texts on arithmetic including use of decimal place value notation. At the first state the western mind could not grasp the full content of zero and avoided the Hindu System. The Arabic word *'sifr'* for *śūnya* was taken as Latin *cifro*. French *chiffre,* and German *Ziffe.* From such words arose a word *'cipher'* in the 14th century. The word cipher is applied for secret writing. Moreover it represents any numeral. Whatever, it may be, *'cipher'* was considered not a suitable word for Hindu *śūnya*. Later, Italian considered another word in Arabic for *śūnya* as

Zefirum and they converted it in their language as *Zefiro* and *Zevero*. However the word zero in Venetian dialect was accepted in 17th century.

The Hindu number system consists of ten numerals including zero. These are sufficient to express any number. *Brahmagupta* defines zero as sum of positive and negative quantities equal in magnitude as $a+(-a) = 0$. He also defines $a+0 = 0$ $-a+0 = -a$ $0+0 = 0$. Zero has three principal characteristics :

1. Absolute value : When representing numerical value other than any positive or negative value. It arises the necessity in the representation when a number is subtrated from itself i.e. $a-a = 0$

2. Positional notation : With zero, the position of different numbers are ascertained to give actual value of the number as in 100 or 1000

3. Place holder or spacer : In abacus a blank space was kept in some position where no number arose and it was kept blank like 7 8. Zero fills the blank and gives the numerals true value of number as 708.

The Hindu numeral system is a great milestone in mathematics. It is true that all the nations builts their notational system but the same was bereft of zero. Without zero, they could not proceed far in the computational field as also mathematical concepts. As soon as they adopted The Hindu Zero in their fold it gained new life and made progress in mathematics.

Beyond the normal notations for expressing numbers the Hindu developed word numerals and alphabetical numerals to suit the authors

for writing. Both the systems were developed with the concept of the fundamental decimal place-value numerals.

Sabda saṁkhyā Praṇāli

The Hindu followed an unprecedented system to express number. They searched for mathematical contents of words and derived many words to bear the characteristic of the number, though in an implicit manner. It was the practice with the Hindu authors to write in verse. Even the mathematical texts were written in the same style. Among them some had the poetic mind, so that poetry and mathematics flowed in a new wave. But in this atmosphere they were to write many numbers and mathematical expressions which did not fit in the verses. Thus they adopted a new style of expressing numbers through word-numerals. These words were again arranged in a systematic way so that the whole set of words represented number in place-value system.

Bhāskarācārya gives his year of birth in the following language

Rasaguṇa pūrna mahi sama sakabhupa samaye 'bhavan momoṭpaṭṭih. Though word numerals were freely used in order to distinguish it from literary work probably the number sequence was written back-wards. Here the first word is *'rasa'* which means taste. Since the days of *Śusṛuta* six typs of tastes were identified as *amla* (sour) *madhura* (sweet) *tikta* (bitter) *lavana* (salt) *kasaya* (astrinqent) and *katu* (pungent). Thus *rasa* denotes 6. This number can also be denoted by *ṛtu* (season), *rāga* (six principal modes of Indian classical music) *sāstra* (six Hindu scriptures). The second word *'guna'* signifies three primary qualities as mentioned in Gita. They are *saṭṭva, rajas*, and *tamas*. *Trikal* (past, present and future) *loka* (heaven, earth, hell) stand for three. The word *pūrna* which means full, represents zero. *Ākāsa* (sky) *vyoma* (air, ether) also signify zero. The fourth word 'mahi' stands for one as there is a single earth. *Śaśi* (moon) *ādi* (first) *bhu* (earth) have the power to represent one. We thus arrive at a figure 6301 and writing in

correct formit denotes 1036 saka year or 1114 AD or 1115 AD according to month. As the same is not available, the year 1114 AD is accepted. *Bhāskāracārya* has expressed at the age of 36 he composed *Śiddhānta Śiramoṇi*. He used the words *rasa guṇa* to denote the year. Thus in 1150 AD the book was written.

Beauty, rhythm and knowledge go side by side with the selection of such words. Moreover the author has the opportunity to select a particular word to suit his purpose from a wide variety of words. It has already been mentioned *Śaśi* (moon) represent one. If the word does not suit, he can search for other names of moon also as *indu*, *vidhu*, *candra*, śitāṁsu. Further he must be acquainted with mathematical content of a word. The word *karaṇīya* (that which ought to be done) stands for 5. which are *ahiṁsā*, *Sunṛta*, *asṭeya*, *brahmacharya* and *aparigraha*. Though word numerals for 0 to 9 are sufficient to express numbers, yet for rhythmical arrangement of syllables into verses and for denoting a number word beyond 10 may be in demand. In such cases also a single word expresses a larger number. Thus dik, dis (directions) *aṅguli* can show 10. *Tithi* (lunar cycle), *Pakṣa* (fortnight) are well known for 15. Every human being has 20 nakha or nails, thus there is no difficulty in using the word for 20. Howmany teeth one has ? The answer 32 permits one to use danta for 32. Again a word *jagati* which represents Vedic rhyme, stands for 48. Forty nine assumes the word *tāna* which stands for keynotes of a song. Further the same number can be represented in various ways.

1.	Kha	- guṇa	- kara	- ādi
	0	3	2	1
2.	kha	- loka	- karṇa	- candra
3.	ākāsa	- kāla	- netra	- dharā.

Thus the science of word - chronograms is built and this brought a new type of knowledge. The reader is face to face with the knowledge

that the text he is reading may contain a word which in fact represents a number. He must know there are 32 teeth, lunar period is 15 and key-notes of vedic songs as 48. In creating such knowledge the role of tradition is most vital. It cannot be taught in a school. The only way is to read large number of texts and try to grasp the inner meanings of such words. This is certainly a notable invention of the Hindu mathematicians. Thus literature and mathematics are joined together for new applications. Thus a very large number is represented by the system *kha* (o) *kha* (0) *asta* (8) *muni* (7) *rãma* (3) *asvi* (2) *netra* (2) *aṣṭa* (eight) *śara* (5) *rātripāh* (1) give 1,582,237,800.

Akṣara saṁkhya praṇāli

Āryabhaṭa's alphabetic system of notation is another innovation of expressing large numbers. The associated word are generally very cryptic and not to be found in a dictionary. Strangely the rule and the use are written by him one after the other in his book. He describes the rule as :

Vargāksarāni varge' varge' *vargākṣarāni kāt ṅmau yaḥ khadvinavake savrā* nava *varge' varge navāntyavargavā*.

Datta & Singh translates thus : "Thus varga letters beginning with k (are used only) in the varga places. the avarga letters in the avarga places (thus) ya equals *ṅmau* (ṅa plus ma) : the nine vowels (are used to denote) the two nines of zeros of varga and avarga (places). The same (procedure) may be (repeated) after the end of the nine varga places. It must be stressed that the system is a modification of an original Hindu system of notation of numbers. In order to use numbers in the form of words in a verse, this is another devise. The system allows one to form very large numbers upto 10^{18}."

The famous grammarian *Panini* is credited as first user of letters of alphabet for denoting numbers. He used some letters in a very restricted

way to signify the verse numbers. The use of word numerals was very successful in signifying ordinary numbers. But in representing a big number a whole line sometimes was required. This idea was not in favour with the authors.

Āryabhaṭa considers letters under two folds *varga* and *avarga*. The 25 letters from *k* to *ma* fall under varga group and eight letters from *ya* to *ha* are included in avarga group. The odd places are denoted by an even power of 10 eg hundredth place is denoted by 10^2. Such numbers are perfect squares. The even places which are denoted by odd powers of 10 is recognised as avarga places. The varga letters will go along with varga places and avarga letters with avarga places. The directive is clear from *"nyāsaśca sthānānām"* 0,000,000.000' which means writing down the places we have 0,000,000.000 and *Āryabhaṭa* directs us to use number by the word *kha*. The alphabetical system is known as *akṣara saṁkhyā praṇāli*. The literal meaning of *akṣara* is that which is indestructible, that which is not subject to loss or extinction.

The Hindu alphabetical system of numerals is entirely different from other systems, in concept, treatment and representation of numerals. While others adopted a straight cut process but many failed to understand the true nature of *Āryabhaṭa's* system.

Following *Śivasūtra Āryabhaṭa* considered 42 letters of which 33 are consonents and 9 are vowels. Ordinarily there are 4 vowels and they are divided into long and short vowels. In this context only short vowels have been considered. However if perchance any long vowel is used, it is to be treated as the same as the corresponding short vowel. Vowels do not represent any number, but being associated with a consonent fix the position in the number system which is again guided by number of zeros followed. The first 25 consonents have values serially from 1 to 25. The next 8 consonents *ya* to *ha* have values 3 to 10. Incidentally *ga* to *na* also have the same values, but two sets are distinguished separately.

Each of the nine vowels denotes two notational places, one for *varga* letters another for *avarga* letters. Thus 'a' with *varga* letter has the value 1 or 10^0 and with *avarga* letter $10^1=10$. Thus *ga*=3x1=3. *ya*=3x10=30. In the last sequence similarly we have g+ou=3.10^{16} & y+au=3.10^{17}. Some scholars made a mistake in alloting 10 times values to the avarga letters to ease the problem but this goes against *Āryabhaṭa's* principle. On this basis of his deduction. the following table is prepared.

Vowels	a/ā	i/ī	u/ū	r/r̄	L/L̄	e	ai	o	au
Vowel number	1	2	3	4	5	6	7	8	9
value with varga letter	10^0	10^2	10^4	10^6	10^8	10^{10}	10^{12}	10^{14}	10^{16}
value with avarga letter	10^1	10^3	10^5	10^7	10^9	10^{11}	10^{13}	10^{15}	10^{17}

Now value of ha' is 10, thus with the 9th vowel 'au', it will represent 10^{18} and this determines the highest place which can be represented under the scheme. This is a clever step. One can enjoy a recreational game with the construction of words and finding the values according to the system.

with vowel no3(u)		with vowel no9(au)	
4th letter 'gha; of varga	4th letter (ba) of varga	18th letter da=18 varga	8th letter (ha) ha=10 avarga
↓	↓		
ghu=$4x10^4$	bu=$6x10^5$	dau=$18x10^{16}$	hau=$10x10^{17}$

In order to give sine values of angles from $3^0 45'$ to 90^0 at $3^0 45'$ interval, he created several words out of his alphabetical system. These words are nonsense, not to be found in a dictionary. The values of sine differences are denoted as 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, 7. This is the oldest trigonometrical table of the world.

Decimal place-value Notation

The decimal place value notation is one of the chief characteristics of the modern number system or rather Hindu number system. This charactreistic was observed long ago in *Viṣnu-Purāna*, *Vāyu-Purāna*, *Vyasa-bhasya* of *Patanjali*, *Sārirako-bhasya* of *Sankarācārya*. Vasumitra illustrated the system by his narration. The *Viṣnu Purana observes* : 'O *brahmana*, from one place to the next in succession, the places are multiplied by 10. The eighteenth one of these (places) is called *Parārdha'*. *Sankarācarya* corroborates the same as : Just as, although the stroke is the same, yet by a change of place it acquires the values one, ten, hundred, thousand etc. In the representation of 1111, the same symbol for one has occupied several places and each one is different in value. This is the keynote of place value system.

Let us consider the mathematical content of place-value system. Consider a number 4785. On division by 10 we get divisor as 478 and remainder 5 which occupies the unit's place. On dividing successive quotients by 10 we will obtain 8,7,4 as successive remainders and they occupy ten's hundred's thousand's places. Moreover the actual value of 4 is $4x10^3$, that of 7 is $7x10^2$, 8 as $8x10^1$, and the unit $5=5x10^0$. This can be suitably represented as :

$$4785 = 4.10^3+7.10^2+8.10^1+5.10^0$$

In a general representation of a number denoted by $(x_n x_{n-1} \ldots x_1 x_0)$ where x_0, x_1 are various numerals 0-9. The above when mathematically represented takes the form :

$$x=x_n 10^n+x_{n-1} 10^{n-1} \ldots\ldots + x_1 10^1+x_0 10^0 .$$

In this unique representation the digits x_0 to x_n are remainders when x is continually divided by 10. Clearly the remainders will be any one of 0,1,2,...9. when the divisor divided by 10, leaves no remainder we get o for the particular x_k. The number n shows the number of digits as

n+1. When a number moves to immediate left hand position the power of 10 also increases by one and the number attains a value ten times the previous one.

In the above scheme, we call 10 the base of the number system. Being 10, it is called decimal system. The base can be taken as we like but the formation of number will depend on the value of base(b).

$$y=a_nb^n+a_{n-1}b^{n-1}\ldots\ldots + a_1b^1+a_0b^0 .$$

The principal features of the place-value system can be suitably catagorised as

1. The number of symbols used is equal to the value of the base.
2. One symbol must be zero
3. When a number is multiplied by the base number, each digit shifts to the next left-hand position and the vacant place in the unit place is occupied by zero. This position is privileged one for zero.
4. Each symbol attains its original value in unit place. But for any other placce the value is multiplied by a power of base, whose power depends on number of shifts from the unit place.
5. No separate symbol is required to identify the position in the hierarchy. We may quote here the line from *Vyāsa-bhāsya* on *Yoga-sūtra* of Patañjali : *'Yathā ekā rekhā śatasthāne sataṁ, daśa.sthāne daśa, ekañcaikasthāne'* which needs no translation.

Numerals are a social necessity of man to assess possessions and also measurement. Thus we find development of numerals in every civilizations. The civilizations of Eqypt, Babylon, China and the advanced cultures of *Phoenician*,. Greek, *Hebrew* and *Roman* designed numerals in their ways. *Maya Civilization* has made a distinct mark in this respect. Lastly *Arabic* culture not only inherited Hindu number system with zero but also was responsible for propagation of the same in the western word. In this respect they gained a very high honour as the system was known as Arabic numerals and later and presently *Hindu-Arabic*

numerals, though it is admitted they have no contribution in this discovery.

It will be interesting to make a quick survey on how far the numerals developed under the above civilizations and cultures truly belonged to the place-value critereon. Many historians of mathematics have described some of them as place-value system.

Among the various numeral-system majority are linked to the decimal. The symbols are adopted either in multiples of 10 or simple alphabetical numerals. No system includes zero and there was no provision to denote if same number is subtracted (345-345). The abacus system which prevailed for normal computation, kept a blank space for the position where no explicit number arose due to computation as 5 6 for 506. The gap may or may not have any meaning according to reference.

The numerical system of the *Egyptian Civilization* consists of a single vertical stroke for one and pictographic symbols, six in number, for each power of 10. from 10 to 10^6. They used to write from right to left, i.e. least significant figure to most significant figure in the system. Each symbol was repeated upto 9 times to represent 90,900 etc. This was the practice, but they could write any symbol in any place, by which the represented number suffered no change. In order to understand the value of such representation one has to calculate from the symbols used. The system fails to meet the conditions of place-value system. There was no base, no zero, no limitation of symbols as for reprenting 5000, 5 symbols of 1000 are used.

Extreme economy has been shown in the *Babylonian* number system. Only two numerals, one in the shape of wedge stands for one and a symbol like two wings of a bird for 10. simultaneously they followed decimal and sexagesimal system. Thus Y may represent one of 60, 60^2 etc. moreover YYY may represent 3 or 60^3+60^2+60. As a small sign denotes as a place holder where no number arises, some prefer to call the system as place-value system. Actually there is no zero and the system does not conform to the essentials of place-value system.

The early Chinese used counting rods made of bamboo, wood, ivory or bone. The representation of numbers are alternately horizontal and vertical for 1-9, 10-90, 100-900, 1000-9000. The use is reflected in abacus, where place-value system is followed but in the absence of zero, the empty space could not be suitably represented. The later symbols are formed in two groups, first group with symbols of 1 to 9 and the second group for 10,100,1000. To represent 3000, it requires two symbols, one for 3 another for thousand as 三 千. Thus for 3256 seven symbols will be required.

The *Phoenician* used 1, v, x, L,C,D for 1,5,10,50 100 and 500. They wrote from left to right, with symbol representing most significant highest number being placed in the extreme left. To represent 8, 4 symbols are required as VIII. No zero is required for signifying numbers, as the concept is included in the symbols. The total symbols are to be evaluated to infer the actual value. Here change of position of a symbol does not change the value of representation. It does not belong to place-value system. At a later period the *Romans* adopted the same pattern.

The *Greek* observed alphabetical system of numerals in groups as 1-9, 10-90, 100-900 with 27 letters. Devices were made to represent 1000 marks. These are self designated symbols and are not affected due to change of position. Later following the principles of Phoenecians, they built a new series of numerals : 1 Γ Δ H X M for 1, 5, 10, 100, 1000, 10,000. An economy measure of numerals was adopted by use of the symbol for 5 with other symbol to represent 50,500,5000 as Γ^{Δ} = 50 instead Δ Δ Δ Δ Δ. There was no necessity of zero in the system. The system does not adhere to the critereon of place-value system.

The *Hebrew* alphabet consists of 22 letters. They are utilised to represent 1-9, 10-90, 100-400. With special symbols created by Juxtaposition of two symbols, upto 900 could be written. Thus it is not a place-value system.

With dash and dot, *Maya Civilization* was able to create 19 symbols from 1-19. They followed vigesimal system i.e. base 20. A single

dot one represents one while $\overset{\cdot\cdot\cdot\cdot}{\equiv}$ represents 19. Thus they follow the first critereon of the place value system as there are 19 numerals and an eye like pictograph for place holder. But this symbol is not similar to other number-denoting symbols. Thus total number of symbols for numerical representation falls short by one. Again a single symbol of dot represents one and a dash 5. All other numerals upto 19 are composed of requisite number of dots and dash as in the case of 19. They were in difficulty to assign the number of days in a year as 360 as 20 base gives $20^2=400$. They made a deviation in the number system in the 3rd place as 18x20=360; otherwise 20 worked like a base in other places. Thus the system can be taken rough place-value system, as they also devised an eye o like Pictograph as a place-holder which did not enter the computation to be recognized as zero.

The original alphabetical system of the Arabs followed the *Hebrew* system. Thus numbers in groups 1-9, 10-90, 100-900 were represented by letters of the alphabet. At that stage no zero appeared. However with the cultural development in the land they copied the Hindu system of numerals. Though the symbols for 1-9 are in *Arabic*, the symbol for zero is retained from the *Hindu* system. Though *Arabian* writes from right to left, but in writing number they follow *Hindu system*- left to right.

The idea of base system helps one to understand clearly about the formation of numbers. The same has been described before. The base of a number can be suitably changed to meet special purpose. The *binary system* has become very popular due to its use in computers. Computers also use *octal* (8) and *hexadecimal* (16). The number of numerals required for a particular base is zero and base number minus one. Thus for *binary system* 1 and 0 are sufficient. For *octal* o and numbers 1 to 7 are required. Any number with a base can be converted to any other base. The concept has been extended to *nega-binary* system where the base is-2 (minus two) : It aims to represent both positive and negative numbers by positivie and negative numbers by positive numbers of base-2.

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THE FAN PROBLEM AND ITS HISTORY IN THE JAPANESE MATHEMATICS

S. TANAKA *, A. KUBO** AND N. SATO***

ABSTRACT

There is a bloc-printed book *Mathematics Summary* (*Sangaku Teiyo,* see Photo.1) in K. ISEKI's collection of books. The book has an Appendix, which has four problems. We are interested in the first problem in the Appendix. The answer is to seven places of decimals. Kiyoshi ISEKI (1919-live, Emeritus Professor of Kobe University and naruto University of Education) asked us whether its solution and answer were correct.

We have investigated Problem 1 (see Photo.2). Its solution is correct, and its answer is shown to seven places of decimals, which agrees with seven places of decimals of a true value. We are surprised at its correctness.

Key Words : Sangaku Teiyo, Shinko TAKEDA, Wasan, the cosine theorem, the formula of double angle, Pythagorean theorem.

Photo.1

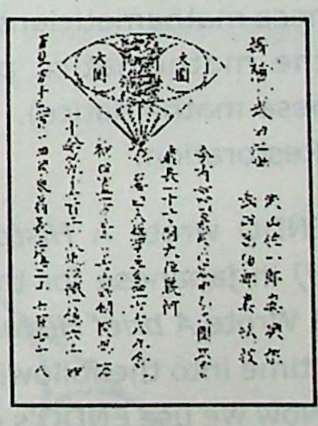
Photo.2

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I. Introduction

[1] Aim of this paper

In this article, we shall make sure of the answer and solution of Problem 1 in the *Mathematics Summary*. The book was published at first in 1833, and its author is Shin·oh JIKU. According to Toshisada ENDO, this publishing was in the Fifth Period (1772-1868) of the hsitory of the Japanese Mathematics.

[2] This book and the author

Shingen TAKEDA was a pupil of Masanobu SAKA. TAKEDA studied mathematics at the Saijo School, then established the foundation of the Takeda School and died in 1846. Shin·oh JIKU was a pupil of Shingen TAKEDA and a Buddhist in Kawachi Province, and wrote *Mathematics Summary (Sangaku Teiyo, 3 vols.)* in Japanese. Problem 1 in study was represented in this book and solved by Shinko TAKEDA, and was refereed by Shinzi TAKEDA, who were both sons of Shingen TAKEDA.

[3] A history of problems written in a fan

Since the Meiji Restoration, Japan imported the western civilization rapidly and made progress on every side. Japanese mathematicians paid little attention to the "Wasan" i.e. the mathematics peculiar to Japan (or Traditional Japanese mathematics). The Wasan had existed before the Meiji Restoration.

Toshisada ENDO wrote A *History of Japanese Mathematics (Dai Nihon Sugaku Shi)* in japanese for the first time. For the second time, Tsuruich HAYASHI Wrote *A brief History of the Japanese mahematics*. T. ENDO divided the time into the following five periods, and HAYASHI used ENDO's division. Now we use ENDO's division we shall be concerned with geometry chiefly.

1) The First Period (from the earliest age to 553 A.D.)

During this period, the Traditional Japanese mathematics was not influended by the Chinese mathemat-

ics. It seems that perimeters, areas and volumes of geometrical figures were measured. These measurements were geometrical operations for practical use but not for theoretical.

2) **The Second Period (554-1591)**

For the first time, the Chinese mathematics was imported. Since this period, counting rods called Sangi had been employed in practical calculations. The simplest case of Pythagorean theorem, in which the hypotenuse is 5 when the other two sides are 3 and 4, was proved.

3) **The Third Period (1592-1672)**

The Chinese mathematics was again imported. The famous mathematician Takakazu SEKI introduced several mathematical methods and established the foundation of the Seki School. Since this period the Japanese people have been using a sort of abacus called "Soroban" as the instrument of practical calculations.

Purely geometrical problems appeared in *Sanpo Ketsugishoh, 5 vols* published in 1659 by Yoshinori ISOMURA. For example, see the following problem :

Problem (Sanpo Ketsugishoh). As in Fig. 1, three circles are inscribed in a rectangular triangle whose two sides are known : Determine the radii of the circles.

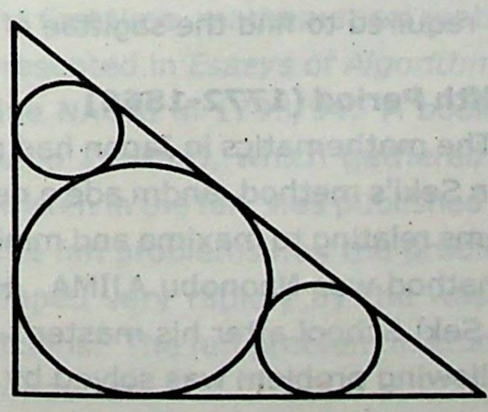

Fig.1

Mitsuyoshi YOSHIDA, occupying a very important position in the history of the Japanese mathematics, wrote a book entitled *Jinkoki* in 1627, which was the second mathematics book published in Japan and which was very popular.

4) **The fourth period (1673-1771)**

The mathematics in Japan had the great progress in this period. That is to say, the Seki School reached its highest level. Yet the Traditional Japanese mathematics did not reach its highest level. Katahiro TAKEBE, SEKI's proper successor, compiled a book entitled "Taisei Sankei" Containing the most profound principles of the Seki School. The chief problem is as follows :

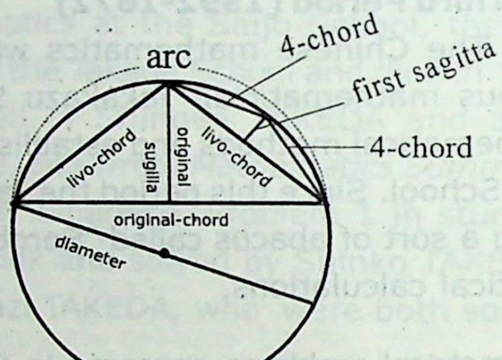

Fig.2

Problem (Taisei Sankei). As in fig.2] "two-chord" are inscribed in a circular segment. Then "four-chord", and so on. Required to find the respective sagittae, that is : The original arc is bisected and then bisected again, and so on; required to find the sagittae of the respective arc.

5) **The Fifth Period (1772-1868)**

The mathematics in Japan had progressed more strik ingly in Seki's method, andm ade a new method of solving the problems relating to maxima and minima. The inventor of this new method was Naonobu AJIMA. He became head-master of the Seki School after his master's death in the year 1772. The following problem was solved by AJIMA.

Problem : As in Fig.3, divide a circular segment into two parts by a sagitta and let a square and a circle be inscribed in each part.

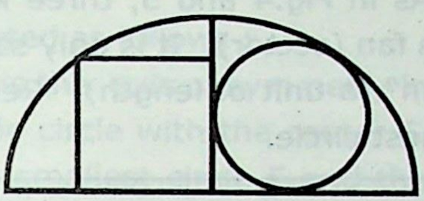

Fig.3

Then the chord and the sagitta of the circular segment are known, and besides, chord+sagitta+side of the square+diameter of the circle and

$$\frac{\text{sagitta}}{\text{chord}} + \frac{\text{diameter of the circle}}{\text{sagitta}} + \frac{\text{side of the square}}{\text{diameter of the circle}}$$

are known : To find the side of the square and the diameter of the circle.

One day AJIMA happened to see this problem written on a board and put on the wall of a famous shrine in the city of Kyoto. It was a custom at that time to hang a tablet "San Gaku", on which a mathematical problem and its solution were described, in a Shinto shrine or a Buddhist temple. AJIMA solved G.Malfatti's problem in 1789. Their solutions are independent of each other.

6) **Appearing of the fan problem**

For the first time, mathematical problems written in the fan were presented in *Esseys of Algorithms (Sanpo Zuihitsu)* by Masamine NAGAI in 1793/94. *A book of Five Algorithms (Gomei Sanpo Zenshu)*, which gathered only mathematical problems written in the fan, was published by Yoshiyuki IESAKI in 1814. The fan problems i.e., the problems written in a fan were developed very rapidly by the wasanka i.e., Japanese mathematicians. The fan problem in study appeared in 1833 and was wonderful.

Let us finish writing the history of the Wasan and return to the proper problem.

II. Problem, answer and solution by Shinko TAKEDA

Problem 1. As in Fig.4 and 5, three kinds of circle are described in a given fan (sector). It is only said that the radius of the fan is 10 "sun" (a unit of length). Required to find the diameter of the largest circle.

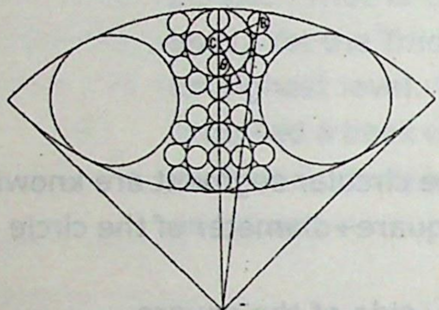

Fig.4

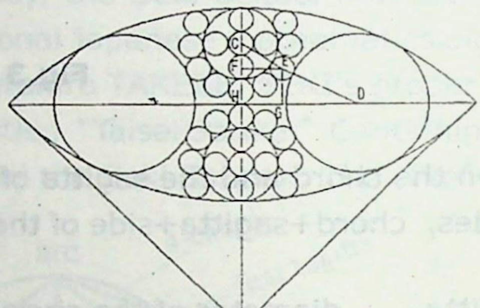

Fig.5

(These two figures are drawn by a personal computer "NEC 9801-VX2" on the values of our process.)

Afterward units of length are omitted. The answer is as follows : The diameter of the largest circle is $4.3417789+\alpha$, where α is smaller than 10^{-7}.

The solution is as follows : Subtracting the product of 189728 and the square root of 13074 from 164011914, multiplying the difference by the radius of the fan, dividing it by 327787557, you get the diameter of the largest circle. That is,

$$\frac{164011914-189728\sqrt{13074}}{327787557}.10.$$

As required.

III. Our process

[1] How to interpret Fig.4 and 5

Fig. 4 is interpreted as follows :

Int.1 The four smallest circles are inscribed in a middle circle crosswise.

Int.2 A line "m" is the axis of symmetry in Fig.4.

Int.3 The number of each kind of circles is just as shown is the Fig.4

15104OE

Int.4 The middle circle with the center C and the smallest circle with the center B are both inscribed in the fan.

Fig.5 is interpreted as follows ;

Int.5 A line "n" is the axis of symmetry in Fig.5.

Int.6 The middle circle with the center C and the smallest circle with the center E, the smallest circle E and the largest circle with the Center D are circumscribed with each other respectively. And further the largest circle is inscribed in the fan.

[2] **Our solution and answer**

Let "X", "a", and "b" be the radii of the largest circle, the fan and the smallest circle respectively. Since, by Int.1, the smallest circles are inscribed in the middle circle crosswise, we see that the radius of the smallest circle is one third of the middle circle. In Fig.4, applying the cosine theorem to an isoscales traingle with a vertical angle θ, we have

$$\mathrm{Cos}\ \theta = \frac{(4b)^2+(4b)^2-(2b)^2}{2.4b.4b} = \frac{7}{8}$$

Using the formula of double angle two times, we obtain

$$\cos 2\theta = 2(\cos\theta)^2 - 1 = \frac{17}{32}$$

and

$$\cos 4\theta = -\frac{223}{512} \ .$$

In fig.4, by Int.4, the smallest circle B is inscribed in the fan. Therefore,

AB = a—b.

Similarly the middle circle C is inscribed in the fan. Therefore,

AC = a—3b.

Furthermore the middle circle C and the smallest circle B are circumscribed with each other. We obtain

BC = 4b.

By Int.3, $\angle ACB=4\theta$. Applying the cosine theorem to ΔABC, we have

$$(a-b)^2 = (4b)^2 + (a-3b)^2 - 2\,(4b).(a-3b)\cos 4\theta.$$

From the last equation it follows that:

$$(1) \qquad b = \frac{11}{289}\,.\,a\,.$$

In Fig.5, by Int.6, the largest circle D is inscribed in the fan. Therefore,

$$AD = a-x.$$

By Int.6, the middle circle C and the smallest circle E are circumscribed with each other. Therefore,

$$CE = 4b.$$

By Int.6, the largest circle D and the smallest one E are circumscribed with each other.

$$DE = b+x.$$

Draw an orthogonal from D to a line segment AC. Let the foot of the orthogonal be H. Similarly, draw orthogonals from E to a line segment CH and From E to DH respectively. Let the feet of the orthogonals be F and G respectively. From $\angle ECF = 2\theta$, we obtain

$$CF = \frac{17}{8}\,.\,b \quad \text{and} \quad FE = \frac{7\sqrt{15}}{8}\,.\,b.$$

From $EG = FH = CH - CF$ and $CH = 5b$, applying the Pythagorean theorem to ΔEGD (by Int.2 and Int.5),

$$GD = \sqrt{DE^2 - EG^2} = \sqrt{X^2 + 2bx - \frac{465}{64}\,.\,b^2}$$

Lastly, from $HD=HG+GD$, applying the pythagorean theorem to ΔADH (by Int.2 and Int.5), we have

$$AD^2 = HD^2 + AH^2$$

and

$$(a-x)^2 = \left[\frac{7\sqrt{15}}{8}.b + \sqrt{x^2+2bx - \frac{465}{64}.b^2}\right]^2 + (a-8b)^2.$$

Substituting (1) for the last expression and arranging the result, we have

(2) $1363505a-5548800x=77\sqrt{15}.\sqrt{5345344x^2+406912ax-56265a^2}$

In order to solve (2) for x, squaring both sides of (2), we have

$$94730603973\ x^2-47399443146\ ax+5825468165\ a^2=0.$$

Solving this quadratic equation for x, we have

$$x = \frac{82005957\pm 94864\sqrt{13074}}{327787557}.a.$$

As easily seen, the plus sign is not suitable. Therefore, the largest circle's diameter 2x is as follows :

$$2x = \frac{164011914-189728\sqrt{13074}}{327787557}.a.$$

This result agrees with the solution of Sangaku Teiyo perfectly. Computing the above expression (a=10) to nineteen places of decimals (using "UBASIC" and "μ-MATH" : computing softwares for numerical calculation, and using a personal computer : "NEC 9801-VX2"), we have

$$2x=4.341778969719999079$$

Summary

As noted previously, the anser of *Sangaku Teiyo* is as follows :

$2x=4.3417789+\alpha, \quad 0<\alpha<10^{-7}.$

This value agrees with our value until seven places of decimals. It is surprised that Shinko TAKEDA computed the value precisely by "Soroban" i.e., a sort of abacus.

References

[1] **Toshiada ENDO** : *A Historyof Japanese Mathematics (Dai Nihon Sugaku Shi),* enlarged ed., 1981, from original (1893).

[2] **Tsuruichi HAYASHI** : *A Brief History of the Japanese Mathematics,* Nieuw Archief voor Wiskunde, Tweede Reeks, Deel VI, 1905, Deel VII, 1907.

[3] **Tsuruichi HAYASHI** : *Collected papers on the old Japanese Mathematics (Wasan Kenkyu Shuroku),* 1985 (reprint), from original (1937).

[4] Japan Academy : *A History of the Japanese Mathematics before the Meiji* (Meiji Zen Nihon Sugaku Shi, Ist vol. (1954), 2nd vol. (1956), 3rd vol. (1957), 4th vol. (1959), 5th vol. (1960)).

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VEDĀṄG IDENTITY OF MATHEMATICS IN INDIA

S. A. PARAMHANS *

Chronologically Mathematics in ancient India can be viewed as follows :

1. Vedic Period (antiquity-1200 B.C.) : The most remarkable contributions of this period are the numerals, and simple operations.

2. Vadāṅga period (1200 B.C.-500 B.C.) : During the Vedāṅga period the contributions were mostly in the field of geometry and astronomy.

3. Infant or Dark Period (500 B.C. - 500 A.D.) : Darkness of this period is coined due to the fact that except for Jaina's Gaṇitanuyoga works and Bakṣālī manuscript, no other Hindu work belonging to this period is found now. However, *Chandaḥ Sūtras* of *Piṅgal* used mathematical zero and in the treatises *Sūrya Prajñapati* and *Candraprajñapti*, which belong to *Gaṇitānuyoga* of the Jainas, we find some important contributions, ellipse being one of them. It is noteworthy here that the credit of inventing ellipse has been wrongly attributed to the Greece geometer Manacmus (350 B.C). But prior to this, it is mentioned in *Sūrya Prajñapti* (500 B.C.) and *Dhammasaṅganī* (400 B.C.). Dhammasaṅganī uses the term Parimaṇḍala for it, which has also been used in *Śatapatha Brāhmana*[1] (6.7) and *Bhagwatī Sūtra* (Jaina work of 300 B.C.) where it has been classified in two ways namely *Pratara Parimaṇḍala* (place ellipse) and Ghana parimaṇḍala (elliptic cylinder).

MATHEMATICS AS VEDĀṄGA

The following verse of *Vedānga Jyotiṣa* is quoted very often in support of the fact that Mathematics was held with a high esteem in

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ancient Hindu culture :

yathā Śikhā Mayūrāṇām nāgānām maṇayo yathā,
Tadvadvedāṅgaśās trāṇām gaṇitam mūrdhani sthitam[2].

Obviously, we can draw two conclusions from it as follows :

(a) Ganita was considered as one of the *Vedāṅga-śāstras*;

(b) It was holding the highest position among the vedāṅga-śāstras.

It should be noted here that the vedānga (limbs of veda) were aimed to help in a proper understanding, interpretation and application of the vedic hymns. These were[3] *sikṣā* (Phonetics), *kalpa* (Ritualistics), *Vyākaraṇa* (Grammer), *Nirukta* (Etymology), *Chandas* (prosody) and *Jyotiṣa* (Astronomy and Astrology). *Sikṣā* was considered as nose, *Vyākaraṇa* as mouth, *Chandas* as legs, *Nirukta* as ear, *Kalpa* as hands and *Jyotiṣa* as eyes of the personified Veda[4]. The first, third, fourth and fifth limbs were meant for appropriate recitation and interpretation of the vedic hymns while kalpa gave rules and regulations regarding vedic rituals and provided code of conduct for the society. Appropriate time for the rituals and for various private and public affairs was decided by *Jyotiṣa* which also indicated the astral motions and their consequences at a particular time on any individual or mass.

Remarkably, the above list of vedāṅgas given by Muṇḍakopanisad, does not seem to include Gaṇita. In this connection, it should be noted here that *in and upto the period of Vedāṅga Jyotiṣa there was an integrated subject which was sometimes called Gaṇita and sometimes Jyotisa. So "Gaṇitam mūrdhani, Sthitam" is quite worthwhile.* But in the later vedic period when the integrated subject *Jyotiṣa* or *Gaṇita* developed, it was divided, as is witnessed by *Chāndogya Upaniṣad*[5], into three branches, namely *Rāśi vidyā* (Arithmetic), *Nakṣatra vidyā* (Astronomy) and *Daiva vidyā* (Astrology). Later on, Rāsi vidyā initiated the process of iṣṭakarma which, when developed, took the form of *Kuṭṭaka Avyakta Gaṇita* or *Bījagaṇita*.

As has been said, Kalpa dealt with rules and methods for performing rituals and ceremonies. it was also divided into three categories as Śrauta, Gṛhya and Dharma. *Śulba sūtras* from the subcategory of *Śrauta Sūtras* and dealth with the geometrical mensurations and constructions related to fire altars. In the later period, *śulba* was separated from *kalpa*. So, in the later vedic period, *Pāṭīgaṇita* (Arithmetic) and *Bījagaṇita*, which were separated from *Astronomy*, joined with *śulba Vijñāna* (geometry) (which was separated from Kalpa) to form an independent discipline 'Mathematics' of the later times. It consisted of the following subtopics : *Parikarma* (fundamental operations vyavahāra Practical arithmetic), *Rajju* (geometry), *Rāśi* (rule of three), *Kalāsavarṇa* (fractions), Yāvattāvat (simple equations), Varga Samīkaraṇa (quadratic equations), *Ghana Samīkaraṇa* (cubic equations), *Varga-varga* (biquadratic equations), Vikalpa (permutation and combination). It is indicated by the Jaina's Sthānāṅga Sūtra (350 B.C.) as :

Parikamman Vavahāro Rajjū Rāsī Kalāsavanne ya,
Jāvantāvati Vaggo Ghano Tataho Vaggavaggo Vikappota[6].

(*Sūtra* 747)

Informations can also be found from *Arthaśāstra*[7] of *Kauṭilya* belonging to the period of Candragupta Maurya (322 B.C.). Here we find the problems dealing with taxation, debt and interest, partnership, barter and exchange etc. .

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REFERENCES

1. Śatapatha Brāhmaṇa : Ed. Ramanath Dikṣit ; Chaukhamba Sanskrit Sansthan Varanasi (1984).

2. Lagadh : Yājuṣa Vedāṅga Jyotiṣa; Indian Journal of History of Science 19 (1984) Appendix, verse 4.

3. Muṇḍakopaniṣad; Motilal Banarasidas (1964); 1.1.5.

4. Pāṇinīya Sikṣā : Ed. Badari Narayan Pandey; Sudarasana Book Agencies, Varanasi (V.S. 2046) verses 41, 42.

5. Chāndogya Upaniṣad : Motilal Banarasidas (1964) 7.1.2., 7.2.1.

6. B.B. Datta and A.N. Singh : History of Hindu mathematics ; Part I, Motilal Banarasidas, Lahore (1935), P.8.

7. Kauṭilya : Artha śāstra. Ed. & Tr. by R. Shamsastry Bangalore (1991).

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PAÑĆARĀŚIKĀDAU - THE INDIAN GOLDEN RULE COMPOUND

with special reference to Bhāskara's Līlāvatī and its commentaries

V. MADHUKAR MALLAYYA *

ABSTRACT

While *Trairāśika* and *Vyasta Trairāśika* are effective tools for handling problems involving simple ratio and proportions, those involving Compound Proportion are dealt with the Rule of Compound Proportions of Rule of Odd terms such as *Pañćarāśika*, *Sapta rāśika*, *Navarāśika* etc. according as the number of terms involved in the computation are five, seven nine etc. The knowledge of *Trairāśika* can be traced as far back as the period of he *Vedānga Jyotiṣa* and this rule perfected by the Indians was highly appreciated and held in high esteem all over the world. Rule of Five, Seven etc are compound Rules of Three and a detailed explanation of the Rule of Compound Proportions along with illustrations can be found in various Indian works. This paper deals with the Rule of Five and so forth with special reference to *Bhāskara's Līlāvatī* and its commentaries like *Buddhivilāsini, Kriyākramakarī* etc. The commentaries provide a vast literature on the Rule and they contain several important enunciations and illustrations from earlier authorites, thus throwing light on the knowledge possessed by the Indians regarding the Rule of Odd terms and its relation with the Rule of Three.

Key Words

Trairāśika, Pañćarāśika, Saptarāśika, Navarāśika, Pramāṇa Pakṣa Ićchā Pakṣa, Bahurāśi Vadha, Svalparāśi Vadha.

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Trairāśika (Rule of Three) is the rule for computing a wanted result from three given quantities generally termed *Pramāṇa, Pramāṇa phala* and *Icchā* where *Pramāṇa* and *Icchā* are of same denomination and *Pramāṇa phala* is of another denomination. The rule is termed *Trairāśika* because of the involvement of the three quantities in the computation. The rule is used when the ratio of two quantities is equal to the ratio of two other quantities that correspond to them. Among these four quantities, three are known and one is required. When the ratio of two quantities is equal to the reciprocal of the ratio of the two quantities that correspond to them, then another rule called *Vyasta Trairāśika* is used. *Trairāśika* is a basic rule in arithmetic and it occurs in all Indian mathematical works. The rule is found to have been stated in the *Vedānga Jyotiṣa*, [R-VJ 24, Y-VJ 42] where it is given as a rule for computing the wanted result from three given quantities categorised as *Jnāna* (the known) and *Jneya* (to be known) which correspond to the terminology *Pramāṇa pakṣa* and *Icchā pakṣa* of later times and the history of the rule can be traced as far back as the period of *Vedānga Jyotiṣa*, if not earlier. The use of fractions like *Kala* (16th part) and *Śapha* (8th part) as interest on debt is found in the *Atharva Veda* [VI. 46.5] and for the calculation of interest on loan, a basic knowledge of the Rule of Three is necessary. The knowledge of the rule is also a prerequisite in various determinations in connection with the constructions of *vedic* altars such as in the determination of number of bricks of specific area to occupy an altar of prescribed area etc. [1] All these facts suffice to prove the Hindu knowledge of the rule during the period of *vedic* literature.

According to the Rule of Three direct,

$$\text{Icchā phalam} = \frac{\text{Pramāṇa phalam X Iccha}}{\text{Pramāṇa}}$$

This simple rule having vast application in the science of computations was prefected in India and if found its way to Europe through the Arabs [2]. This rule was highly appreciated and held in high esteem all over the world. In the west it was often called the *'Golden Rule'* for its excellency and according to Hodder of 17th cent, it was commonly and rightly called by this name for *'as gold transcends all other metals, so doth this rule all others in Arithmetik'* [3].

While the Rule of Three and Inverse Rule of Three are effective tools for handling all problems involving simple ratio and proportions, those involving compound proportion are dealt with the Rule of Compound Propertions or Rule of Odd Terms such as the Rule of Five, Rule of Seven, Rule of Nine, Rule of Eleven etc, according as the number of terms involved in the computation are five, seven, nine, eleven etc. In Sanskrit they are termed in general as *'Panćarāsikādau'* (Rule of Five and so forth). *Panćarāśika* is the Rule of Five, *Saptarāśika* is the Rule of Seven, *Navarāśika* is the Rule of Nine, etc.

According to *Bhāskara* II, in the method of five, seven, nine or more terms, the fruits and divisors are transposed and the product of the larger number of terms when divided by the product of the smaller number of terms gives the quotient which will be the quantity sought [4]. For illustrating the Rule, *Bhāskara* gives the following examples in the *Līlāvatī* [5].

Example 1 (For Rule of Five)

If the interest on 100 for 1 month is 5, then say what will be the interest on 16 for one year. In the same way find the time from the principal and interest given and also tell what the principal sum will be, from the time and produce given,

Statement for finding interest :	1	12	
	100	16	Answer : Interest is $9^3/_5$
	5		
Statement for finding time :	1		
	100	16	Answer : months 12
	5	48/5	
Statement for finding Principle :	1	12	
	100		Answer : Principal 16
	5	48/5	

Example 2

If the interest on 100 for $1^1/_3$ months is $5^1/_5$ then tell what the interest willbe on $62^1/_2$ for $3^1/_5$ months.

Statement	4/3	16/5	
	100	125/2	Answer : interest $7^4/_5$
	26/5		

Example 3 (for Rule of Seven)

If 8 best quality silk scarfs of length 8 cubits and breadth 3 cubits cost a 100 *niṣkas*, then tell quickly trader, if you know trade, what the cost will be for a similar scarf of length 3½ cubits and breadth ½ a cubit.

Statement :	3	$^1/_2$	
	8	$^7/_2$	Answer : 0 *niṣka*, 14 *dramma*, 9 *paṇa*, 1 *Kākiṇi*, $6^2/_3$ *varāṭaka*
	8	1	
	100		

Example 4 (For Rule of Nine) :

If 30 benches, 12 *aṅgulas* thick, 16 cubits wide and 14 cubits long cost a 100 *niṣkas*, then comrade, tell me what the cost of 14 benches will be, which are 4 less in every dimension.

Statement :	12	8	
	16	12	Answer : *Niṣkas* 16 $^2/_3$
	14	10	
	30	14	
	100		

Example 5 (for Rule of Eleven) :

If the cart hire for conveying the benches of the dimension first mentioned in the above example through a distance of 1 gavyuti is 8 drammas, then say, what the hire will be for conveying the second set of benches four less in every dimension through a distance of 6 gavyutis ?

Statement :	12	8	
	16	12	Answer : *drammas* 8
	14	10	
	30	14	
	1	6	
	8		

According to the commentator *Gaṇeśa*, the Rule of Five, Seven, Nine etc are Compound Rules of Three comprising of two or more sets of three terms. *Sūryadāsa, Parameśwara, Śankara*, Damodar Misra and L. Jha have also described the Rule of Odd terms as comprising of two or more Rules of Three in their respective commentaries on the *Līlāvatī*. Thus the commentators explain the Rule of Five as Double Rule of Three, Rule of Seven as Treble Rule of Three, Rule of Nine as Quadrauple Rule of Three, Rule of Eleven as that comprising of five Rules of Three and so on. Damodar Miṣra has generalised this by stating that Rule of 2n+1 terms will be comprising of n Rules of Three. In the *Kriyākramakarī, Śankara* supports the argument (that the Rule of Odd terms comprise of definite number of Rules of Three), by quoting the remark made by *Bhāskara* I of 525A.D. in the *Āryabhaṭīyabhāsya* viz; 'The combination of two Rules of Three will become Rule of Five. The *āćārya* (*Āryabhaṭal*) has described

only the fundamentals of proportion; from that fundamental Rule of Proportion, all others such as Rule of Five and so forth follows, for Rule of Five etc consists of union of two or more Rules of Three, Rule of Five being union of two Rules of Three, Rules of Seven being three Rules of Three, Rule of Nine being four Rules of Three and so on' [6].

Regarding the interpretatin of the world *cid* used by *Bhāskara* II in the statement of the rule, there is some difference of opinion among the commentators. According to *Gaṇeśa*, *cid* ralates to denominators of fractions and the transposing of divisors is indeed right because in division involving fractions, the rule of inverted divisor has to be applied [7]. According to *Śankara*, if fractionals are involved, then all the denominators of the *Iccha pakṣa* are transposed to *Pramāṇa pakṣa* and those of the *Pramāṇa pakṣa* to the *Iccha pakṣa*. The underlying principle in this is that in the division of fractions *'the numerators* and *denominators of the multipliers and divisors should be multiplied by one another'* as advocated by *Āryabhaṭa*I [8]. This underlying principle behind the transposition of *cid-* (divisors) of the *Pramāṇa pakṣa* to the *Icchā pakṣa* and that of *Icchā pakṣa* to the *Pramāṇa pakṣa* has been pointed out by *Śankara* suported by citations from the *Āryabhaṭīya, Brahmasputa Siddānta,* and *Pāṭīgaṇita*. According to *Brahmagupta*, 'In the case of Rule of Odd terms from Three to Eleven, the required result is obtained by transposing the fruits from one side to the other and then dividing the product of the larger number of terms by that of the smaller. If there be fractionals, it must be known that the denominators of the fractions on both sides are to be transposed *(jneyamiha teṣu bhinneṣūbhayatas'cheda samkramaṇam)* [9]. According to *Śrīdhara*, after transposing the fruits from one side to the other, and then having transposed the denominators *(chedānām vyatyāsam kṛtva)* multiply the numbers (so obtained on either side) and divide (the product) on the side with larger number of quantities by the other [10]. Thus both *Brahmagupta* and *Śrīdhra* have

prescribed the transposition of denominators when fractions are involved. Thus quoting from earlier authorities, *Śankara* explains the term *ćid* as relating to denominators of fractional quantities involved. Like *Gaṇesa* and *Śankara*, Damodar Misra has also explained the process of transposition of *ćid* with the help of the rule of inverted divisors for division of fractions [11]. According to P.K. Koru [12] if there are fractionals on either side, then their denominators are to be transposed; if there be a fraction on one side and a whole number on the other side, then the whole number is to be written with a denominator unity and this unit denominator on one side and the denominator of the fraction on the other side are to be transposed. P.K. Koru has suggested this (ie. the idea of putting 1 for denominator of the whole number on one side corresponding to a fractional on the other side) just to avoid the confusion that might arise while identifying the *'larger set of terms'* and the *'smaller set of terms'* after transposing the *phala* and *ćid*. Thus Koru has also used the term *ćid* in the sense of denominators of fractions. However Ramakrishna Deva and *Sūryadāsa* have a different opinion. *Sūryadāsa* refers to the *Gaṇita Kaumudi* in support of his view as follows : "There are two sets of terms; those which belong to *Pramāṇa* and those which belong to *Iććhā*. The fruit in the *Pramāṇa* set is called *Pramāṇa phala* and that in the *iććhā* set is called *'ćid'*. They are to be transposed, or reciprocally brought from one side to the other side, i.e, put the phala in the *iććhā* set and the *ćid* in the *Pramāṇa* set. Would it not be enough to say, transpose the fruits of both sets ? The author of the *Gaṇita Kaumudi* replies : The designation of divisor serves to indicate, that, after transposition, the fruit of the second set, being included in the product of the multiplication of the less set of terms, the product of the greater set is to be divided by it. Some, however interpret it as relative to fractions, but that is wrong : for the word would be superflous" [13]. Even though this explanation is vague, it is clear that *Sūryadāsa* relates term *'ćid'* to the *Iććhā phala* sought. After transposition of this *'ćid'* and *Pramāṇa*

phala, the greater set will contain the *Pramāṇa phala* which is known and the product of the smaller set X *ćid*

= Product of the greater set from which it follows that the *ćid* or *Ićchā phala* sought

$$= \frac{\text{Product of the greater set}}{\text{Product of the smaller set}}$$

Sūryadāsa has however erred in making the remark that the interpretation of *ćid* as *divisors* of fractional quantities by others is wrong. The validity of both the interpretations may be checked by writing the quantities of the *Pramāṇa pakṣa* and *Ićchā pakṣa* in algebraic terms.

In the case of Rule of Five,

Pramāña pakṣa	*Ićchā pakṣa*
P_1	i_1
P_2	i_2
f	c=?

$$\frac{P_1 P_2}{i_1 i_2} = \frac{f}{c} \text{ and so } P_1 P_2\, c = i_1.i_2.f$$

$$\text{From this, } c = \frac{i_1.i_2.f}{P_1 \cdot P_2}$$

where is $i_1.i_2.f$ is the product of the larger set of terms and $P_1.P_2$ is the product of the smaller set of terms after transposing the known Pramāṇa phala and the unknown *Ićchā phala* c. Thus the explanation of the term *ćid* given by *Sūryadāsa* is correct. In case fractions are involved say

$$P_1 = \frac{a_1}{b_1} \quad P_2 = \frac{a_2}{b_2} \quad ,i_1 = \frac{x_1}{y_1} \quad ,i_2 = \frac{x_2}{y_2}$$

$$\text{then, } c = \frac{\frac{x_1}{y_1} \cdot \frac{x_2}{y_2} \cdot f}{\frac{a_1}{b_1} \cdot \frac{a_2}{b_2}} = \frac{x_1 . x_2 . f . b_1 . b_2}{y_1 . y_2 . a_1 . a_2}$$

Here the numerator is the product of the larger set of terms and the denominator is the product of the smaller set of terms after transposing the fruits and the divisors of the fractions and thus the interpretation of the term *ćid* by the other commentators as relating to division of fractionals involved is also correct.

The commentators have given detailed explanation of the two sets of terms viz; *'Bahurāśi pakṣa* and *'Svalparāśi pakṣa'. Bahurāśi Pakṣa* is the most numerous and *Svalparāśi pakṣa* is the less numerous. According to *Gangādhara*, *Bahurāśi pakṣa* is that to which the fruit *(Pramāṇa phala)* is brought [14]. According to *Gaṇeśa*, if there be fruit on both sides, then that in which the fruit of requisition is brought is the *Bahurāśi pakṣa* [15]. *Parameśwara* explains *Bahurāśi vadha* as the product obtained by multiplying the *Phalarāśi* (fruit) with the product of the *Iććhā rāśis* and the *Svalparāśi vadha* as the product of *Pramāṇa rāśis* only [16]. According to *Śankara* also *Bahurāśi vadha* is the product of all the *Iććhā rāśis* together with the *Phala rāśi* and *Svalparāśi vadha* is the product of all the *Pramāṇa rāśis* [17]. Damodar Misra has explained how the number of terms of *Iććhā* side exceed the number of terms on the *Pramāṇa* side, Since there are as many *Iććhā rāśis* as there are *Pramāṇa rāśis*, the number of quantities on both sides are now equal. But when the *Phala rāśi* is placed in the *Iććhā* side,the number of quantities on the *Iććhā* side will exceed the number of quantities on the *Pramāṇa* side by one term. So the product of the *Iććhā rāśis* together witht he *Phala rāśi* will be the product of the larger set of terms called *Bahurāśi* vadha and the product of *Pramāṇa rāśis* will be the product of the smaller set of terms called the *Svalparāśi vadha* [18]. P.K. Koru has also explained like

this. From the explanations of the terms *Bahurāśi vadha* and *Svalparāśi vadha*, *Bhāskara's* formula for *Iććhā phalam* may be put in the form

$$\textit{Iććhā phalam} = \frac{\textit{Bahurāśi vadha}}{\textit{Svalparāsi vadha}} = \frac{\text{Product of } \textit{Iććhā rāśis} \times \textit{Phala rāśi}}{\text{Product of } \textit{Pramāṇa rāsis}}$$

Detailed working of the five exmaples given by *Bhāskara* can be found int he commentaries by Ramakrishna Deva, Gaṇeśa, Mahīdhara, Parameśwara, Śankara, P.K. Koru and L Jha. Apart from these five examples, Śankara has given several additional examples in the Kriyākramakarī to demonstrate the vastness of the Rule [19]. Unlike the other commentators, *Ramkrishna Deva* has used the *apavartana* principle to abridge the process by reducing the terms on both sides by removing their common divisors. This helps to make the computation very simple. The detailed working of *Bhāskara's* first example as given by the commentators may be given as follows :

The two sets are

1	12
100	16

Transposing the fruits, the sides are

1	12
100	16
	5

product of the larger set = 12 X 16 X 5 = 960

product of the smaller set = 1 X 100 = 100

So the required quantity is 960/100 = 48/5

To find the time, the two sides are

1	
100	16
5	48/5

ie,

1	
100	16
5/1	48/5

(as suggested by Koru, the denominator of 5 is taken as 1)

Transposing the fruits,

1	
100	16
48/5	5/1

& transposing the divisors

1	
100	16
48	5
1	5

Product of the larger set = 1 X 100 X 48 X 1 = 4800

Product of the smaller set = 16 X 5 X 5 = 400 and so the required period = 4800/400 = 12 months.

To find the principle amount, the two side are

1	12
100	
5	48/5

Transposing the fruits, the two sides are

1	12
100	
48	5
1	5

Product of the larget set = 1 X 100 X 48 X 1 = 4800

Product of the smaller set = 12 X 5 X 5 = 300

∴ Required Principal amount = 4800/300 = 16

Using the *apavartana* principle as suggested by Ramakrishna Deva, in the Manoranjana, the above example can be worked out as follows. The two sides after transposing are

1	12
100	16
	5

Abridging both sides by removing the common divisors 4 and 5, the two sides are

1	3
5	16
	1

Product of the larger set = 3 X 16 X 1 = 48

Product of the smaller set = 1 X 5 = 5

Hence the required interest = 48/5

To find the time, the two sides (after transposing fruits & divisors and

abridging by removing the common divisors 5,5, and 16) are

1	.
4	1
3	1
1	1

Hence *Bahurāśi Vadha* = 1 X 4 X 3 X 1 = 12

Svalparāśi vadha = 1 X 1 X 1 =1

∴ Required time = 12/1 = 12 months

To find the principal, the two sides (after transposing fruits and divisors and abridging by 5,5,12) are

1	1
4	
4	1
1	1

∴ *Bahurāśi vadha* = 1X 4 X 4 X 1 = 16

Svalparāśi vadha = 1 X 1 X 1 = 1

Hence the required principal = 16/1 = 16

In algebraic terms, the first part of the example canbe put in the form 100 : 16 X 12 : : 5 : x where x is the required interest.

This gives

$$x = \frac{16 \times 12 \times 5}{100} = 9\,{}^{3}/_{5}$$

Similarly the other results can be obtained by putting the required quantity as x. For the second example, the two sides are

4/3	16/5	Transposing fruits	4/3	16/5
100/1	125/2		100/1	125/2
26/5				26/5

Transposing the divisors

4	16
5	3
100	125
2	1
	26
5	1

(100 being a whole number, its denominator is taken as 1 and the denominator of the unknown *Icchā phala* is also taken as 1 as suggested by P.K. Koru)

Hence *Bahurāśi vadha* = 16 X 3 X 125 X 1 X 26 X 1 and

Svalparāśi vadha = 4 X 5 X 100 X 2 X 5

$$\text{Therefore required reuslt} = \frac{16 \times 3 \times 125 \times 1 \times 26 \times 1}{4 \times 5 \times 100 \times 2 \times 5} = 7\ {}^{2}/_{5}$$

Algebraically if x is the required result, 4/3 X 100 : 16/5 X 125/2 :: 26/5 :x

$$\text{and } \therefore \quad x = \frac{{}^{16}/_{5} \times {}^{125}/_{2}}{{}^{4}/_{5} \times 100} \times {}^{26}/_{5} = 7\ {}^{2}/_{5}$$

For the example of Rule of Seven, the two sides are

3	1/2
8	7/2
8	1
100	

Transposing fruits,

3	1/2
8	7/2
8	1
	100

Writing with denominator 1 where ever necessary as stated by Koru, the two sides are

3/1	1/2
8/1	7/2
8	1
	100

Trnsposing the denominators

3	1
2	1
8	7
2	1
8	1
	100

Bahurāśi vadha = 1 X 1 X 7 X 1 X 1 X 100 = 700

Svalparāśi vadha = 3 X 2 X 8 X 2 X 8 = 768

∴ Required result = 700/768 = 0 *niṣka*, 14 *dramma*, 9 *paṇa*, 1 *Kākiṇi*, 6 $^2/_3$ *varātaka*

Algebraically if the required quantity is denoted by x, then

3 X 8 X 8 : 1/2 X 7/2 X 1 :: 100 : x

$$\text{which gives } x = \frac{1/2 \times 7/2 \times 1}{3 \times 8 \times 8} \times 100 = \frac{700}{768}$$

For the example of Rule of Nine, the two sides are

12	8
16	12
14	10
30	14
	100

Transposing the fruits,

12	8
16	12
14	10
30	14
	100

Product of the larger set = 8 X 12 X 16 X 14 X 100 = 1344000

Product of the smaller set = 12 X 16 X 14 X 30 = 80640

$$\text{Required result} = \frac{1344000}{80640} = 16\ ^2/_3 \textit{ niṣkas}$$

Algebraically if x is the required quantity then

30 X 12 X 16 X 14 X 24 : 14 X 8 X 12 X 10 X 24 :: 100 : x which gives

$$x = \frac{14 \times 8 \times 12 \times 10 \times 24 \times 100}{30 \times 12 \times 16 \times 14 \times 24} = 16\ ^2/_3 \textit{ niṣkas}$$

For the example of Rule of Eleven, the two sides are

12	8
16	12
14	10
30	14
1	6
8	

Transposing fruits

12	8
16	12
14	10
30	14
1	16
	8

Product of the larger set = 8 X 12 X 10 X 14 X 6 X 8

Product of the smaller set = 12 X 16 X 14 X 30 X 1

$$\text{Required result} = \frac{8 \times 12 \times 10 \times 14 \times 6 \times 8}{12 \times 16 \times 14 \times 30 \times 1} = 8 \textit{ drammas}.$$

Algebraically if x is the required quantity, then

30 X 12 X 16 X 14 X 24 X 1 : 14 X 8 X 12 X 10 X 24 X 6 :: 8 : x which gives

$$x = \frac{4 \times 8 \times 12 \times 10 \times 24 \times 6 \times 8}{30 \times 12 \times 16 \times 14 \times 24 \times 1} = 8 \textit{ drammas}.$$

The computation in all these examples can be reduced considerably by proceeding as recommanded by Ramakrishna Deva, by abridging both sides by removing the common divisors and his commentary Manoranjana stands unique in this aspect.

In the *Kriyākramakarī Śānkara* has provided nine more examples, five for Rule of Five, one each for Rule of Seven and Rule of Eleven and

two for Rule of Nine. Examples for Rule of Five are (i) if the interest on 100 for a month is 5, then how much will be that on 60 for a year.

To find interest,

1	12
100	60
5	

Transposing the fruits

1	12
100	60
	5

Product of the larger set = 12 x 60 X 5 = 3600

Product of the smaller set = 1 X 100 = 100

∴ Required Interest = 3600/100 = 36

To find time,

5		36
100	1	60

After transposing fruits, Product of the larger set = 100 X 36 = 3600

Product of the smaller set = 5 X 1 X 60 = 300

∴ Required time = 3600/300 = 12 months.

To find the amount,

1		12
5	100	36

$$\text{Required amount} = \frac{36 \times 100 \times 1}{12 \times 5} = 60$$

To find principal time (*Pramāṇakala*)

5	36	
5	60	12

$$\text{Required result} = \frac{12 \times 5 \times 60}{36 \times 100} = 1 \text{ month}$$

To find *Pramāṇa dhana*

5	36	
1	12	60

$$\text{Required result} = \frac{12 \times 5 \times 60}{36 \times 1} = 100$$

To find *Pramāṇa phalam*

100	60	
1	12	36

$$\text{Required result} = \frac{36 \times 100}{12 \times 60} = 5$$

Śaṇkara has taken this example from *Śrīdhara's Pātīgaṇīta* [20]. Examples parallel to this can be had from *Āryabhaṭīyabhāsya* of *Bhāskara* I [21], *Gaṇitasārasaṁgraha* of *Mahāvīra* [22], *Gaṇita Tilaka* of *Śripati* [23], and *Gaṇita Kaumudi* of *Nārāyana Paṇdita* [24].

(ii) If the interest on $100\ ^1/_2$ for $^1/_3$ month is $1^1/_2$, then what will be that on $17\text{-}^1/_4$ for $7^1/_2$ months

To find interest (*Kalāntaram*), $100^1/_2\quad ^1/_3\quad 1^1/_2\quad 17\text{-}^1/_4\quad 7^1/_2$

Product of *Pramāṇa rāśis* = $^{201}/_6$ Product of *icchā rāśis* = $^{1005}/_8$

Multiplying this by Phala *rāśi*, the product = $^{3015}/_{16}$

$$\therefore \text{Required interest} = \frac{3015/16}{201/6} = \frac{18090}{3216} = 5\ ^5/_8$$

To find *Kālam* (time), $^3/_2\quad ^{201}/_2\ ;\ ^1/_3,\ 5\ ^5/_8,\ 17\text{-}\ ^1/_4$

$$\text{Required result} = \frac{72360}{9648} = 7\ ^1/_2$$

To find mūlam, $^3/_4\quad ^1/_3\ ;\ ^{201}/_2,\ ^{45}/_8,\ ^{15}/_2$

$$\text{Required result} = \frac{36180}{2160} = 17\text{-}\ ^1/_4$$

Similar illustrations can be had from the *Āryabhaṭīyabhāṣya* [25], *Pāṭīgaṇita* [26], *Gaṇitāsarasaṁgraha* [27], *Gaṇita Tilaka* [28] and *Gaṇita Kaumudi* [29].

(iii) If the value of 1 *suvarṇa* of touch 16 is 73, what is that for $1^1/_2$ *suvarṇa* of touch 11 ?

16		11	
1	73	$1^1/_2$	Answer = $75^9/_{32}$ *rūpas*

Similar example occurs in the *Pāṭīgaṇita* [30] and *Gaṇitasārasaṁgraha* [31].

(iv) If with 6 *paṇas*, 8 *droṇas* of rice can be carried through a distance of 1 *Yojana*, then how much will be the expense for carrying a *Khāri* together with a *droṇa* of rice through 3 *Yojnas*.

8		17	
1	6	3	Answer : $\dfrac{6 \times 17 \times 3}{8 \times 1}$ = 38 *Paṇa*, 1 *Kākiṇi*

This example occurs in the *Pāṭīgaṇita* of *Śrīdhara* [32] which has its parallel in the *Gaṇitasārasaṁgraha* [33] and *Gaṇita Tilaka* [34].

(v) If 3 labourers earn 5 rupas in two days, then tell how much will 8 labourers earn in 9 days.

3		8	
2	5	9	Answer : 60 *rūpas.*

Śankara has taken this example from the *pāṭīgaṇita* [35] and similar illustrations can be had from the *Gaṇita Tilaka* [36].

(vi) For illustrating the Rule of Seven the following example is given. If seven blankets with breadth 2 and length 8 fetch 10, then how much will two blankets with breadth 3 and length 9 ?

1	2	10	
2	3		
8	9		Required value = $33\ ^{3}/_{4}$

This example is same as that given in the *Pāṭīganita* [37] and *Gaṇita Tilaka* [38].

(vii) For illustrating the Rule of Nine : If a stone slab having length, breadth and thickness measuring 9,5,1 respectively costs 8, then

what will be the cost of 2 stone slabs of measure 10,7 and 2.

1	2	8
9	10	
5	7	
1	2	Cost of the stone slabs = 49 $^{7}/_{9}$

This appears in the *Pātīgaṇita* [39] and *Gaṇita Tilaka* [40]. A similar example can be had from *Gaṇita Kaumudi* [41].

(viii) This is also for illustrating the Rule of Nine. If an elephant having length 7, breadth 2, and height 6 (cubits) takes 1 *droṇa* of food, then how much will an elephant of length 10, breadth 3 and height 9 cubits take as diet ?

1	1	1
2	3	
6	9	
7	10	Answer : 3 *droṇa* 0 *āḍhaka*, 3 *prastha*, 1 $^{1}/_{5}$ *kuḍava*

This example is also from the *Pātīganita* [42] and similar example can be seen in the *Āryabhaṭīyabhāsya* [43] and in *Pṛthuḍaka Svāmin's* commentary on the *Brahmasputa Siddhānta* (xii. 10-12)

(ix) For the Rule of Eleven : if an elephant of length 9, breadth 8 and height 5 hasts take 17/2 *prasthas* and enjoy roaming carelessly with rut on its temple, then how much will 10 elephants of length 10, breadth 9 and height 7 *hastas* take in 5 days.

9	17/2	10
8		9
5		7
1		10
1		5

Product of *Pramāṇa rāśis* = 360

Product of *icchā rāśis* = 31500

Phalarāśi = 17/2

$$\text{Required reuslt} = \frac{\text{Product of } icch\bar{a}\ r\bar{a}\acute{s}is \times phala\ r\bar{a}\acute{s}i}{\text{Product of } Pram\bar{a}\d{n}a\ r\bar{a}\acute{s}is} = \frac{31500 \times 17/2}{360}$$

= 2 *Khāri*, 14 *droṇa*, 1 *ādhaka*, 3 *prastha*, 3 *kuḍava*

Śankara has thus given an elaborate exposition of the Role of Odd terms by citing examples from earlier works. Influence of *Śrīdhara*'s work *Pātīgaṇita* on *Śankara* can be seen from these additional examples, most of which are taken from the *Pāṭīgaṇita*. The popularity of *Śrīdhara's* work during the period of *Śankara* is evident from this. Moreover the vast literature provided by *Śankara* with several important enunciations and illustrations from the celeberated works of earleir authorities shows that *Śankara* has made a vast study of their works before writing the commentory on *Bhāskara's Līlāvatī. Śankara's* citation from the commentory of *Bhāskara* I on *Āryabhaṭiya* pertaining to the Rule of Odd terms and that from the *Brahmasputa Siddhānta* throws light on the knowledge possessed by the Indians regarding the Rule of Odd terms and its relation with the Rule of Three. While *Bhāskara* I remarks tha the Rule of Odd terms follow from the fundamental Rule of Three by compounding two or more Rules of Three, *Brahmagupta* treats the Rule of Three as a particular case of the Rule of Odd terms.

While the Rule of Odd terms beyond three went by the names *Pancarāsika, Saptarāsika, Navarāsika, Ekādasarāsika* etc in India, names beyond five were rarely used in the outside world. All beyond that for three were commonly tagged under the general name *'Double Rule of Three'* or *'Compound Rule of Three'* or *'Cojoint Rule'* or *'Plural Proportion'* In the European countries the Rule went by the names *'Regvle de cinque parte'* of Ortega (1512A.D), *'Die Regel von fünff zalen'* or Kobel (1514, A.D.), *'Regel Von fünffen'* of Rudolf (1526 A.D), 'Regela Quinque/oder zwyfache Regel dc Try of Thierfelder (1587 A.D.) Some other names are *'Regula Duplex'* of Gemma Frisius (1540 A.D.), *'La Regle double'* of Trenchant (1566 A.D), *Regela Trivm Composita* of Clavius (1583 A.D)

etc. English Mathematician Recorde (1542 A.D) called the Rules beyond three by the name *'Double Rule of Three'* or the *'Golden Rule Compound'*. Finally the name *'Compound Proportion'* emerged which became quite general in the 18th century A.D. [44].

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REFERENCES

1. *Baudhāyana Śulba Sūtra* [5.10-11, 7.2] Translated into English by G. Thibaut, Published by the Research Institute of Ancient Scientific Studies, New Delhi, 1968
2. History of Hindu Mathematics Part I by B.B. Datta and A.N. Singh (p. 210), Asia Publishing House, Bombay, 1935.
3. History of Mathematics, Part I by D.E. Smith (p. 486), Dover Publications, New York, 1958.
4. *Līlāvatī* of *Bhāskarācārya* with *Kriyākramakari* of *Śankara* and *Nārāyana* critically edited with Introduction and Appendices by K.V. Sarma (p 191, st.82), Vishveshvaranand *Vedic* Research Institute, Hoshiarpur, 1975.
5. ibid 4, *Kriyākramakari* (pp 192, 194, 198-200).
6. *Āryabhaṭīyabhāsya* of *Bhāskara* I (*gaṇita*. 26); *Āryabhaṭiya* of *Āryabhaṭa* Critically edited with the commentary of *Bhāskara* I, with Introduction and Appendices by K.S. Shukla, Indian National Science Academy, New Delhi, 1976.
7. *Līlāvaṭī* of *Bhāskara* with *Buddhivilāsini* and *Līlāvatī vivaraṇa* of *Gaṇes'a* and *Mahīdhara*, Ed. by D.V. Apte, Part I, (p 77), Anandasram Series, No. 107, Poona, 1937.
8. *Āryabhaṭīya* of *Āryabhaṭa* (*gaṇita* 27). Critically edited with English Translation Notes, Comments and Indexes, by K.S. Shukla and K.V. Sarma, Indian National Science Academy, New Delhi, 1976.
9. *Brahmaspuṭa Siddhānta* of *Brahmagupta* (xii. 11-12). Edited by Rama Swarup Sarma, Indian Institute of Astronomical and Sanskrit Research, New Delhi, 1966.
10. *Pātigaṇita of Śrīdhara* (*Sutra* 45), Translated by K.S. Shukla, Lucknow University, 1959.
11. *Līlāvatī* of *Bhāskarācārya* with *vāsanā* of Pt. Damodar Misra Edited by D. Jha (p. 75), Mithila Inst. Dharbanga, 1959.

12. *Srī Bhāskarāchārya raćita Līlāvatī* with Malyalam commentary by P.K. Koru (p 94) Published by Mathrubhoomi Printing and Publishing Company, Ltd. Kozhikode, 1954.
13. Algebra with Arithmetic and Meusuration from the Sanskrit of *Brahmagupta* and *Bhāskara* by H.T. Colebrooke (p.35, foot note. 3), London, 1817.
14. ibid 13, H.T. Colebrooke (p. 35, foot note 4)
15. ibid 7, *Buddhivilāsinī* (p.76)
16. *Līlāvativyākhyā* of *Parameśwara*,. Transcript by N. Ananta Krishna Sarma (p.38) Oriental Manuscript Library, Kerala University, Karyavattom, Trivandrum.
17. ibid 4, *Kriyākramakarī*; (p 191)
18. ibid 11, Damodar Misra (p. 75)
19. ibid 4, *Kriyākramakarī*, (pp 195-196, 199, 200-201)
20. ibid 10, *Pātīgaṇita* (Example 39)
21. ibid 6, *Āryabhaṭiyabhāṣya of Bhāskara I,* [ii. 26-27 (i)]
22. *Gaṇitasārasaṁgraha* of *Mahāvīra* (v 33, 41) Edited with English Translation and Notes by M. Rangacharya, Govt. Press, Madras, 1912.
23. *Gaṇita Tilaka* of *Śrīpati* (p 75, vs. 98) Edited by H.R. Kapadia Gaikwad Sanskrit Series, No. 78, Baroda, 1935.
24. *Gaṇita Kaumudi of Nārāyaṇa Paṇdita,* Part I (p.50. lines 18-21) Edited by Padmakara Dvivedi, Govt. Sanskrit College, Benares, 1936
25. ibid 6, *Āryabhaṭīyabhāṣya* (ii. 26-27)
26. ibid 10, *Pātīgaṇita* (Example 40)
27 ibid 22, *Gaṇitasārasaṁgraha*, (v.34)
28. ibid 23, *Gaṇita Tilaka* (p. 76, vs 99)
29. ibid 24, *Gaṇita Kaumudi* Part I, (p 51, lines 6-9)
30. ibid 10, *Pātīgaṇita* (Example 41)
31. ibid 22, *Gaṇitasārasaṁgraha* (v. 35)
32. ibid 10, *Pāṭīgaṇita* (Example 43)
33. ibid 22, *Gaṇitasārasaṁgraha* (v.36)
34. ibid 23, *Gaṇita Tilaka* (p. 78, vs. 101)
35. ibid 10, *Pāṭīgaṇita* (Example 44)
36. ibid 23, *Gaṇita Tilaka* (p. 77, vs 100)
37. ibid 10, *Pātīgaṇita* (Example 45)
38. ibid 23, *Gaṇita Tilaka* (p.78, vs 102)
39. ibid 10, *Pātigaṇita* (Example 46)
40. ibid 23, *Gaṇita Tilaka* (p. 79, vs. 103)
41. ibid 24, *Gaṇita Kaumudi* Part I, (p. 52, lines 2-5)
42. ibid 10, *Pātīgaṇita* (Example 47)
43. ibid 6, *Āryabhaṭīyabhāṣya* [ii-26-27 (i)]
44. ibid 3, D.E. Smith, (pp 491-492).

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THE MICROBIAL AMYLASE

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The amylase is the enzyme which can cleave the glucosidic linkages in starch and glycogen. The saliva secreted by the salivary gland in our mouth as a reflex response to the presence of food not only lubricates the food rather break down the starch and glycogen into simple molecules that can be easily assimilated by our body. We realise every day while we continue chewing the loaf of a starchy bread it turns into a sweet liquid. The Chyawan Prash Linctus and several other preparations available in the ancient Ayurvedic literature (Kaushik, 1983, 1988) are the examples of stimulators of the process. The author (1996) has already highlighted its application in the industries of baking and milling, preparation of bear, cereals, confectionary, candy, corn syrup, distilled beverages, flavours, pharmaceutical and clinical products.

It is known that microorganisms use enzymes to process specific biological molecules during metabiolism and growth. Some of the same enzymes can be used to process biological molecules in ways that are valuable in food technology. The food industry exploits these catalysts and often uses fungal enzymes. When used in food processing the enzymes may be indispensable to the formation of final product. The enzymes may improve the quality of the product or may hasten the food proceesing steps.

The enzyme α-Amylase which is of great importance in industrial microbiology and has been examined by recombinant DNA method. The

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sources of -amylase have been mouse salivary, mouse pancreas, human salivary, *Saccharomycopsis fibuligera* and wheat. The recombinant fungal host for all these sources was yeast *Saccharomyces cerevisiae*. The malted wheat, barley, bacteria and fungi are common sources of α-amylase.

The wheat flour used in bakery contains some enzyme activities but it lacks the necessary levels of enzymes that determine some of the functional properties of doughs to carry on the automated manufacturing process. The enzymes which are deficient in doughs are α–amylase and protease, therefore, both are supplemented.

The α-amylase from *Aspergillus oryzae*, *A. niger* and *A. awamori* or *Rhizopus* spp are used to supplement the amylolytic activity in flour. According to Fergus (1969), the thermophilic fungi *Humicola isolens*, *H. lanuginosa*, *H. stellata*, *Malbranchea pulchella* var. *sulfurea*, *Mucor pusillus* and *Talaromyces thermophilus* produce the most amylase. The enzymes elevate the levels of fermentable monosaccharides and disaccharides from 0.5% of the dough to concentrations that activate the yeast. A sustained release of glucose and maltose by added and endogenous enzymes provides the nutrients essential for yeast growth and gas production during panary fermentation. An α-amylase supplementation may be beneficial for another reason. The enzyme degrades starch granules usually present in bread flour much more effectively than does wheat β-amylase. As the white bread flours contain 6.7 - 10.5% damaged starch, addition of enzyme in the baking industry is necessary to maintain the yeast fermentation in such doughts. The α-amylase from *A. oryzae* is preferred for the baking since it is heat labile at 60 to 70^{0}c and thus does not survive the baking process. The thermolability of the said enzyme prevents action on the gelatinized starch in the finished loaf and the production of a soft and sticky crumb.

Amylase supplementation also improves the other aspects of bread quality. The enzyme lowers dough viscosity and influences its softness. The treatment improves the volume, taste, crust and toasting qualities. The volume of the bread is improved probably because α-amylase reduces the viscosity of the gelling starch, allowing greater expansion during baking before protein denaturation and enzyme inactivation fix the loaf volume. Amylolytic activity also increases the sugar concentration in bread. This outcome is beneficial in many instances as an elevated sugar content in bread is usually preferred by the consumer. Besides its use on blood amylase treatment inhances the desirable quality of rolls, buns and crackers. The fungal amylase is added to doughs in the form of diluted powders and prepacked doses or water-dispersible tablets. The preparations have been added to flours at the bakery or in some countries at the mill itself. Malted wheat or barley can also be used as sources of enzyme by blending with wheat flour at the mill.

The more heat-labile bacterial α–amylase from species of *Bacillus* is also used in baking. Its properties make it suitable for production of coffee cake, fruit cake, brownies, cookies, snacks and crackers.

Dingle and Solomons (1952) found that fungus isolate may produce amylase on an agar medium but not in submerged culture. The thermophiles have been grown at 45°c in Erlenmeyer flasks (50ml medium per 250 ml flasks) and in petri dishes (50 ml medium per 15 cm dish. The medium contained soluble starch, 5 g; yeast extract, 2g; $K_2H\ Po_4$, 1g; mg SO_4. $7H_2O$, 0.5 g; 1 ml. microelement solution and distilled water to make 1 litre. The p^H after autoclaving should be 7.2; and 20g agar is added per litre for surface culture. The fungi are grown in petri dishes of the same medium for 4 days at 45°C for inoculum. Inoculum for the liquid culture flasks consist of one disc (6 mm diameter) of agar and mycelium per flask obtained by using a sterile cork borer. The petri dish cultures were inoculated by transfer needle, placing a small bit of mycelium on agar

medium as near as possible to the disc edge thus allowing a maximum number of daily observations.

The amylase activity of agar cultures can be determined by flooding with Lugol's iodine solution. The non-hydrolyzed starch in the uninvaded agar medium formed a deep purplish-blue complex with the iodine.

The amylase activity of filtrates can be determined by pipetting 0.1 ml of the filtrate into a cavity bordered in a hardened starch assay agar medium in a petri dish by using a sterile cork borer (13 mm diameter). The assay agar medium generally used constitutes soluble starch 10 g; Na_2 HPo4, 2.84 g; Nacl 0.35 g and agar 20 g per litre (pH 6.9). Hansen (1984) adopted a manual iodine starch method to analyse broths from *Aspergillus oryzae* fermentations. The method is based on degradation of starch by the enzyme, and measurement of the residual starch/iodine complex at 570 mm. Calibration graphs are linear for the range 0.01-0.1 amylase units ml^{-1}.

PURIFICATION AND PROPERTIES OF GLYCOAMYLASE

Futatsugi et al. (1993) detected two forms of glucoamylase in the filtrate of a 6-days old culture *Saccharomycopsis fibuligera* which was cultivated by them under aerobic conditions with an oxygen transfer rate of 35×10^{-7} g mol.ml/min. The glucoamylases designated as I and II resemble each other in terms of pH stability, optimum temperature and the effects of various chemicals on enzymatic activity but the molecular weights (55,000 and 56,000 respectively) iso-electric points (6.5 and 6.1 respectively) thermal stability and km values for maltose differed between the two enzymes.

Saccharomycopsis fibuligera IFO - O III was cultured on the medium which consisted of 40 g soluble starch, 12 g yeast extract, 1 g KH_2 PO_4,

0.05 g $MgSO_4$. $7H_2O$, 0.05 g KC1 and 0.5 g $NaNo_3$ in one l of deionized water. Cultures can be grown at 37°c for 6 days in a jar fermentor of the capacity of 20 l but 12 l of medium. the oxygen transfer rate (OTR) $35x10^{-7}$ g mol/ml/min aeration, 1 vvm; agitation 360 rpm, inner pressure 1.0 kg/cm2.

Part of the culture filtrate was used to the purification. Glucoamylase activity can be assayed with soluble starch as a substrate. The reaction mixture used by Futatsugi et al. (1993) containing 2.0 ml of 1% solution w/v of soluble starch in acetate buffer (pH 5.5) and 0.5 ml of the enzyme solution was incubated at 37°c for 15 minutes. The reaction was terminated by the addition of 0.5 ml of a 0.6 N solution of NaoH. The amount of glucose liberated was determined by the Glucose C test (Wako Pure Chemical Industries Co. Ltd., Osaka). One unit of enzymatic activity is defined as the amount of enzyme that liberated 1μ mol of D-glucose per minute under the mentioned conditions. Electrophoresis of the enzyme can be carried out by the method of Davis (1964).

Isoelectric focussing can be carried out on a column (total volume about 110 ml) which having 0.5% of the carrier ampholyte with a p^H range from 5 to 8 at 600 V for 48 h at 4°c. The fractions of 1 ml were collected. Thermal inactivation of enzymes can be analyzed as follows. Two ml of the enzyme solution in 20 m M phosphate lose buffer (pH 7.0) are heated at 45°C or 48°C. At intervals, aliquots are removed and chilled quickly in ice-water and the residual glucoamylase activity is assayed as described above. The purification of the glucoamylases produced by *Saccharomycopsis fibuligera* IFO OIII can be done as follows :

Step 1. 2- Propanol precipitation.
Step 2. Coloumn chromatography on CM cellulose.
Step 3. Column chromatography on DEAE Sephadex A-50.
Step 4. Column chromatography on Sephadex G-100.

Step 1: 2-precipiation : The culture filtrate (600 ml) can be precipitated with 1.5 volume of 2-propanol at-20°c. The precipitate is

dissolved in 50 m M phosphate buffer pH 6.0 and the solution is dialysed against the same buffer.

Step 2 : Column chromatography on CM-cellulose : The sample of glucoamylase from step 1 is loaded on a column of CM cellulose and equilibrated with 50 m M acetate buffer (pH 4.0). The column is washed with the same buffer and then the active enzyme is eluted with a linear gradient of 0 to 0.5 M Nacl in the same buffer. The active fractions are pooled and concentrated by ultrafiltration on a Diaflo YM 10 membrane (Amicon Corporation, Danvers, Krelend).

Step 3 : Column chromatography on DEAE-Sephadex A-50 : The dialysed solution from step 2 was applied to a column of DEAE-Sephadex A-50 equilibrated with 50 m M phosphate buffer (pH 6.0). Elution is carried out with a linear gradient of 0 to 0.5 m Nacl in the same buffer. The glucoamylase of *S. fibuligera* is separated into two types of chromatography on DEAD - Sephadex A-50. The active fractions of glucoamylase is eluted separately were designated glucoamylase I and II and concentrated separately with a Diaflo YM 10 membrane (Amicon) and dialyzed against 50mM acetate buffer (pH 5.5).

Step 4: Column chromatography on Sephadex G-100 : Each solution of the enzyme obtained from step 3 is further purified by gel-filtration on a column of Sephadex G-100 equilibrated with 50 m M acetate buffer (pH 5.5). The resultant active fractions are concentrated by ultrafiltration. The purified glucoamylase I and II exhibit about 5-fold greater specific activities than those of the culture filtrate. Both purified enzymes were reported to be homogenous on disc gel electrophoresis.

The molecular weights of glucoamylase I and II as determined by the *Sephadex* G-100 gel-filtration method of Andrews (1978) and *polyacrylamide gel* electrophoresis in the presence of sodium dodecyl *sulfate (SDS)* are 55,000 and 56,000 respectively. The carbohydrate

contents of glucoamylase I and II which are determined by the phenol sulphuric acid method are 16% and 19% by weight respectively. The limits of hydrolysis of a 2% solution (w/v) of soluble starch by glucoamylases I and II are known to be 85% and 80% respectively. The reason for the low limits of hydrolysis may be due to the high purity of glucoamylase. According to the researches in W.P.C.I.L. and Kumamota Institute of Japan the Km (i.e. michael's contant) values of glucoamylase I and II for maltose were estimated to be 8.3 mM and 2.6 mM. respectively. The isoelectric points (pI values) of glucose I and II were 6.1 and 6.5 respectively. The pH optima for glucoamylases I and II with soluble starch as substrate were found to be 5.5 and 6.0 respectively. The purified glucoamylases I and II are stable in the pH range of 5.0 to 7.0 when incubated at 37⁰C for 5 h. The optimum temperature is 40⁰C for the activity of both glucoamylase I and II toward soluble starch. The effects of salts on various cations such as Mg^{2+}, Ca^{2+}, Ba^{2+}, Fe^{3+}, Co^{2+}, Mn^{2+}, Zn^{2+}, Cd^{+}, Ni^{2+}, Cu^{2+}, Pb^{2+}, Ag^{+}, and Hg^{2+}, (each of 2 mM) and of other reagents like 2-mercaptoethanol, PCMB monoideacetate, N-ethylmaleimide, iodoacetamide, Nan_3, SDS, Urea, Triton X-100, Tween 20 (a detergent) on the activity of glucoamylase have been studied but both glucoamylases are hardly affected by any of these chemicals. The two glucoamylase differ with regard to thermal stability. The glucoamylase II is more stable to heating than glucoamylase I.

सारांश

एमाइलेज एक एन्जाइम है जो मंड और ग्लाइकोजन को सरल अवयवों में तोड़ सकता है। यह एन्ज़ाइम हमारी पाचन प्रणाली का एक अभिन्न अंग है और अग्न्याशय और लाल-ग्रंथि से निकलने वाला बहुमूल्य स्राव है। खाद्य सूक्ष्मजैविकी और उद्योग में इसकी आवश्यकता ने सूक्ष्मजीव वैज्ञानिकों को सूक्ष्मजीवों से एमाइलेज प्राप्त करने के लिए प्रेरित किया। जापान व अन्य देशों में अन्तर्राष्ट्रीय ख्याति के लब्धप्रतिष्ठ एवं मूर्धन्य कुछ वैज्ञानिकों ने इसके शोधन को और भी सरल बना दिया है।

REFERENCES

1. Anonymous. 1957. Manual of microbiological Methods. Mc Graw-Hill Book Co, New York. pp. 1-315.
2. Davis, B.J. 1964. Disc electrophoresis II. method and application to human serum proteins. Ann. N.Y. Acad. Sci. USA, 121 : 404-427.
3. Fergus, C.L. 1969. The production of amylase by some thermophilic Fungi. Mycologia, 61(6) : 1171-1175. Printed in U.S.A.
4. Futatsugi, M. Ogawa, T. and Fukudo, H. 1993. Purification and properties of two forms of Glucoamylase from *Saccharomycopsis fibuligera*. Journal of Fermentation and Bionegineering, 76(6) : 521-523.
5. Kaushik, P. 1983. Ecological and Anatomical Marvels of the Himalayan Orchids. Today and Tomorrow's, New Delhi. pp viii+123+plates 71.
6. Kaushik, P. 1988. Indigenous Medicinal Plants Including Microbes and Fungi. Today and Tomorrow's, New Delhi. pp vii+243.
7. Kaushik, P. 1996. Introductory Microbiology. Emkay Publications, Post Box 9410, B-19 East Krishna Nagar, Delhi. pp viii+344.

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट
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THE HIGHLY USEFUL MUSK MELLOW

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In Ayurveda, natural medicines are used for the treatment of various ailments. In olden days, the mankind was very much introduced to these due to the closeness to plants because of the living in small towns and villages. Nowadays, the living in the cities, the jungle of concrete, has sent the man away from the very friendly plant kingdom. Through this article, we are making an effort to bring the readers close to the medicinally useful plant kingdom by taking the example of the plant Ambrette, *i.e.*, Musk Mellow, the *Lata Kasturi*. Following Sanskrit Shloka tells much about the substance obtained from this plant :

लता कस्तूरिका तिक्ता स्वाद्री वृष्या हिमा लघुः ।
चक्षुष्या छेदिनी श्लेष्मतृष्णावस्त्यास्यरोग हृता ।।

Musk-mellow is a pungent, delicious, juicy substance which is good for vitality of man and mild, good for eyes, cough and thirst and helps in the treatment of mouth diseases.

Musk-mellow is also known as *Mushka dana*. In fact, in the different parts of the country it is known with different names like *Kasturibhenda* in Marathi and *Karpuribend* in Telegu. To scientists it is known as *Abelmoschus moschatus* or *Hibiscus abelmoschus*.

Attaining a height of one to two meters, this plant occurs in comparatively warmer regions of our country. At places it itself grows wild but in many of the places it is cultivated because of variety of its uses. Leaves of this plant are of varying shapes, usually palmately 5-7 lobed, on long petioles, resembling the leaves of the plant of lady's finger. Flowers are bell shaped, shining yellow, with dark violet spot in lower part. The beans are 2.5-3.0 inch long. In the beginning they are

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green but become reddish and ultimately black on ripening. The seeds are brown-black, small, spherical but slightly flat. On rubbing between the palms they produce musk like odour.

Any well drained, fertile soil, is suitable for cultivating the musk mellow. The cultivation is better in comparatively the warmer regions. Before sowing, the soil may be manured with an equal nitrogen, phosphorus, potassium fertilizer. The better results may be obtained by using half the quantity of nitrogen before sowing and remaining half quantity at flowering. Sowing is done in middle of June. Flowering and fruitning starts generally in November. The spacing between the grown up plants should be approximately 1.5 meter. In one hectare land, 1.0 to 1.5 kg. seed is sufficient for sowing.

One irrigation is sufficient before sowing. In a normal monsoon, no further irrigation is required till September. After this, irrigation shall be done at an interval of 15-20 days.

The problem of weeds may be there. They can be removed once in July and again in September. The beans mature in-between first week of December and last week of February. When three-fourth part of bean has become black-brown, the plucking is done. Itching may sometime take place on touching the plant material, so pluckers should cover their hands etc. before carrying out the plucking. Seeds are taken out from collected beans in shed and stored in a shed itself to prevent the loss of essential oils. The yield of seeds is around 10-12 quintals per hectare. Sometimes, The plant gets infected with *Hibiscus mosaic virus* (HMV). Such a plant should be uprooted and destroyed as soon as noticed. For the safety point of view, 0.2% Thiodon 25ED solution or 0.03% Eldrin is sprayed at the intervals of 20 days or so, once the plant has become one month old.

Sometimes, the plant is attacked by red-mite. For this 0.05% solution of Folidol E-605 should be sprayed, but only once.

As already told, the seeds of musk-mellow when rubbed, give the fragrance of musk. The musk like smell is due to presence of fragrant volatile oil in outer part of the seeds. It is used as a precious perfume. To extract this oil from the seeds, either the steam distillation of crushed seeds is carried out or solvent extraction is carried out in alcohol, ether, petroleum ether and benzene etc. Immature seeds should be discarded before carrying out the extraction.

Besides the musk odour volatile oil, the seeds contain several fatty acids and some resins. Fatty acids are removed by dissolving in aqueous alcohol or by formation of lithium and calcium soaps. Separated fatty acids may be used in edible oils and the left out seed meal may be used as a cattle feed.

The shelf-value of isolated essential oil is very satisfactory if it is stored at low temperatures, completely covered and in dark.

Lata Kasturi has variety of uses. The seeds and musk oil have three kinds of uses. In perfumes, in cosmetics, and in medicines. The mild wine like smell in oil gives typical attractive fragrance in perfumes. If used alongwith rose, neroli and sandal, it gives an excellent fragrance. This makes use of musk mellow oil in creams, lipsticks, hair oils and talcum powders. It is also used to provide an appealing odours to eau-de cologne and lavender soaps.

The oil is useful as a direct cosmetic also. The mixture of crushed seeds with milk is a direct cosmetic material. This mixture is good for curing itch.

In fact, the musk-mellow is used for treatment of different diseases in different countries. In U.S.A. and other western countries it is used as a specific antidote for the venom of the rattle snake. In the Indian subcontinent too, it is used for this purpose sometimes. In Guinea, the seeds are considered to be stomachic and a tonic. In Brazil, the herb is

used as a foementation and an enema. In Philippines, it is used for urinary calculus. In Arab countries the ambrette seeds are used as a coolant, an aphrodisiac, and for stomach disorders.

The powder of musk-mellow seeds is good for eyes. The oil cures the diseases due to 'Kapha' and 'Vata' intestinal complaints. It is good in diseases of the heart. It allays thirst and checks nausea.

A dose of 25 to 50 g is given for treatment of cough, bronchial problems and in fever. A chewing of a couple of seeds keeps the mouth fresh and acts as an appetizer.

The veneral diseases like gonorrhoea are also treated by this plant. A paste consisting of the crushed roots and leaves in sugarcandy is considered to be effective in it.

Recently the experiments are being carried out to establish the use of *Lata Kasturi* in Aromatherapy.

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INCIDENCE OF POVERTY IN INDIA-ITS ESTIMATION AND RELATED DATA GAPS *

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Introduction

Removal of poverty and improvement in the standard of living of the masses have remained the basic objectives of the Indian Planning. These are being achieved through planned economic growth and target oriented poverty alleviation programmes for the poor. To help formulate effective schemes for poverty alleviation, measurement of poverty is essential. Though there is difference of opinion among the experts on the methodology to be adopted for its measurement, the importance of quantification of poverty is well recognised. This paper describes the methodology so far used by the Planning Commission for estimation of proportion and number of poor since 1973-74 and the available data source used for this purpose. The official methodology recently adopted by the Government on March 11, 1997 based on the recommendations of the Expert Group (1993) has also been described and the state-wise estimates of proportions and number of poor as per the new methodology has been presented. Gaps in data required for the estimation of incidence of poverty have also been enumerated.

Defining the poverty line

The question of defining poverty line was first method by the Indian Labour Conference in 1957. A definition of poverty in the Indian Context, was attempted for the first time by a distinguished Working Group set up

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* Views expressed in the paper are those of the authors and not necessarily of the organiation to which they belong.

by the Planning Commission, Government of India in July, 1962. The working Group comprised Prof. D.R. Gadgil, Dr. B.N. Ganguli, Dr. P.S. Loknathan, Shri M.R. Masani, Shri Ashok Mehta, Shri Pitambar Pant, Dr. V.K.R.V. Rao, Shri Shriman Narayan and Shri Anna Saheb Sahashrabudhhe. After taking into account the recommendations of the nutrition Advisory Committee of the Indian Council of Medical Research (ICMR) in 1958, regarding the balanced diet, the Working Group came to the view that the national minimum for each household of 5 persons (4 adult consumption units) should not be less than Rs. 100 per month at 1960-61 prices or Rs. 20 per capita. It further suggested that for urban areas this figures should be raised to Rs. 125 per month per household of Rs. 25 per capita to cover the higher prices of the physical volume of commodities on which the national minimum is calculated. By implication, this meant that the corresponding amount in the rural areas would work out to Rs. 18.90 per capita.

Dandekar and Rath (1971) used an average calories norm of 2250 kilo calories per capita per day for both rural and urban areas as a criterion to define the poverty line so as to segregate the poor from non-poor. On the basis of National Sample Survey (NSS) data on consumption expenditure, the study revealed that an average monthly per capita expenditure of Rs. 14.20 in the rural areas and an average monthly per capita expenditure of Rs, 22.60 in the urban areas both at 1960-61 prices would suffice to meet the requisite calories requirements. Sukhatme (1977, 78) argued that average calories requirement does not represent the minimum below which a person can be treated as undernourished. Bardhan (1971), Rudra (1974), Minhas (1969) and others forwarded different estimates of incidence of poverty at regional Delhi level following mainly the national norms.

Planning Commission constituted a 'Task Force on Projections of Minimum Needs and Effective Consumption Demand' in 1979 to

recommend a poverty line. The methodology as formulated by the 'Task Force' has since then, been used by the Planning Commission for estimating the incidence of poverty in India. There is no unique approach for estimating the poverty line in conformity with the absolute concept of poverty. However, a consensus emerged in the seventies.

The poverty line has been defined by the 'Task Force' (1979) as that expenditure level, which meets the average per capita, per day calories intake of 2400 kilo calories for rural areas and 2100 kilo calories for urban areas. The average calorie norms were estimated by the 'Task Force' on the basis of age-sex- activity specific calorie allowances recommended by the Nutrition Expert Group, 1968. These were averaged using projected age-sex-occupation structure of population for 1982-83 as per the estimates of the Expert Committee on Population Projections, 1977 as weighting diagram. The census occupational structure (1971) and the participation rates based on usual activity status derived from National Sample Survey (NSS) Employment Survey data contained in the 27th Round (1972-73) were also used. The monetary equivalent of these norms (i.e. poverty lines) have been worked out using the 28th round (1973-74) NSS data relating to private consumption both in qantitative and value terms. Using appropriate conversion factors, the calories content of consumption baskets corresponding to various expenditure classes have been worked out. Applying inverse linear interpolation method to the data on average per capita monthly consumption expenditure and the associated calories content of food items in the class, separately for rural and urban areas, it has been estimated that, on an average Rs. 49.09 per capita per month satisfied a calories requirement of 2400 kilo calories per capita per day in rural areas and Rs. 56.64 per capita per month satisfied a calories requirement of 2100 kilo calories per capita per day in urban areas both at 1973-74 prices. The poverty line so estimated implies that having this amount, on an average, an individual will distribute his expenditure between food and non-food items in such a

way that the calorie content of his food consumption satisfies the desired calories norm. Thus, the concept of poverty line used here is partly normative and partly behavioural. This way of deriving the poverty line, while being rooted in a 'norm' of calorie requirement, does not seek to measure the nutritional status and more specifically the incidence of malnourishment or under-nourishment in the population. It focuses rather on the purchasing power required to meet the specific calorie intake standard with some margin for non-food consumption needs. Moreover, the calorie norms satisfy an average and not the minimum required for biological existence, given that there is a considerable variation in the calorie requirement of individuals depending upon the workload, age, sex and occupational structure.

UPDATING THE POVERTY LINE

The poverty cut off points, as estimated above, need updating over time to take care of the changes in the price level. This can be done in two ways :

(i) The poverty line can be updated by using relevant price inflators weighted by appropriate consumption basket,

(ii) A fresh poverty line can be calculated using the data from relevant consumption expenditure survey adopting appropriate calorie norm.

Alternatives indicated above have somewhat different implications for the concept of poverty line and its measurement over time. alternative (i) would amount to defining the poverty line in terms of certain 'purchasing power' with reference to a particular point of time or base year. The poverty line is updated over time to protect this 'purchasing power'. This purchasing power enables the households to buy the requisite number of calories. How and to what extent this purchasing power is actually put to use is not relevant. Alternative (ii) on the other hand is concerned more with the consumption of the requisite calories rather

than the protection of a fixed purchasing power. Over time, consumption patterns across the expenditure classes may change in such a way that the households shift their preferences towards food items which are perceived to be of a better quality though with less calorie content. in such a case updation of poverty cut-off points by protecting the consumption of requisite calories may not bring about any reduction in the incidence of poverty, even though there may have been a substantial increase in the real purchasing power of the households. Thus so long as there is a change in the consumption basket of households due to shifts in individual preferences, the methods of updating poverty line discussed above would give different results.

As per the recommendation of the 'Task Force', Planning Commission used alternative (i) mentioned above to update the poverty line in the Sixth Five Year Plan. The same was continued for the Seventh Five Year Plan. However, instead of wholesale price index which was used for updating the poverty line in the Sixth Five Year Plan, the private consumption deflator was used for the same purpose during Seventh Five year Plan.

ESTIMATION OF PROPORTION AND NUMBER OF POOR

Using the updated poverty line and the data on the size distribution of population by expenditure classes from the households consumption survey conducted by National Sample Survey Organisation, for the reference year, the number and proportion of persons below the poverty line are estimated. The poverty estimates are made separately for rural and urban areas and at national and state levels, using appropriate consumption distributions. In estimating the State level incidence of poverty, the national calorie norm and the corresponding all-India poverty line have been applied on the State specific household consumption distribution, separately for rural and urban areas, quinquennially since 1973-74.

The poverty line so estimated for a given year, aims at estimating the purchasing power, at current prices, required to meet the expenditure associated with the assumption basket that satisfied the calorie norm in the base year 1973-74. The poverty line so defined can be characterised by the facts that (i) as pointed out earlier the base year (1973-74) poverty line is partly normative and partly behavioural as it is based on a certain specified calorie requirement of the population from the NSS consumption distribution for that year. This, however, is not true about the poverty line for the subsequent years. The estimation of the poverty line for the base year and its mere updating thereafter amounts to holding the consumption basket implicit in the poverty line and which meets the normative requirement constant over time. In the process, ignoring changes in the consumption expenditure pattern of the population, thereby freezing the behavioural aspect of the poverty line. Thus, for any year after 1973-74 so long as individual has the requisite purchasing power to afford the reference base year consumption basket even though he/she chooses to actually spend on a different basket which may fetch him/her lesser calories than the required calorie norm, the individual would be classified as non poor, (ii) besides accounting for the calorie intake, the poverty line makes a certain allowance for non-food consumption including clothing, footwear, fuel and light and other miscellaneous goods and services though not necessarily the respective minimum, (iii) the calorie norm anchoring the poverty line is an average for a refercne group and not the minimum required for the biological existence. Its use to measure the incidence of undernourishment in the population would, therefore, be grossly inappropriate, (iv) the use of poverty line so defined for estimating the incidence of poverty at the state level presupposes that the price vector of the consumption basket implicit in the estimated all India poverty line for 1973-74 and its movement over time is identical across all states and for rural and urban areas. Clearly this is restrictive assumption if the objective is to estimate incidence of poverty by states.

It has been observed that the national total of household consumption expenditure as estimated on the basis of the result of NSS household estimated in national Accounts Statistics (NAS). To make the estimates of total private consumption expenditure consistent from both the sources the expenditure levels reported by the NSS is raised by a factor of proportion, capturing the differences between the total private consumption as obtained from NSS and the total as estimated by NAS. This factor is applied uniformly to all expenditure classes. The incidence of poverty is then estimated with the adjusted distribution of consumption expenditure. The estimates of poverty based on the aforesaid procedure are presented in table 1 below :

Table 1 : Estimates of poverty (All India)

		1972-73	1977-78	1983-84	1987-88
Poverty line (Rs.)	Rural	41.0	60.0	101.8	131.8
(at current prices)	Urban	47.0	69.9	117.5	152.1
Proportion of }	Rural	54.1	51.2	40.4	33.4
People below }	Urban	41.2	38.2	28.1	20.1
Poverty line }	Combined	51.5	48.3	37.4	29.9

Source : Planning Commission (1993), Government of India.

There has been significant decline in the incidence of poverty over years. The decline in the incidence of poverty has been achieved due to combined effect of faster economic growth and poverty alleviation programmes.

OFFICIAL METHODOLOGY FOR ESTIMATION OF PROPORTION AND NUMBER OF POOR.

The aforesaid procedure for estimation of poverty has been criticised and its limitations have been pointed out from a number of angles. Broadly they fall in two categories, the first related to the

concept itself and the second arising from the data and methodologies used in India for estimating the poverty line. Planning Commission, Government of India constituted in September, 1989 an Expert Group to consider methodological and conceptual aspects of estimation of proportion and number of poor in India. The Expert Group submitted its report in July, 1993 and its recommendations have broadly been accepted by the Government of India on 11th March, 1997 which henceforth would be the official basis for quantifying the proportion and number of poor in India, Main features of the official methodology for estimating proportion and number of poor on basis of the recommendations of the 'Expert Group on Estimation of Proportion and Number of Poor (1993)' are as under.

(i) Conceptually, the Expert Group has relied on the poverty line approach recommended by the 'Task Force' as being used by the Planning Commission for estimation of the extent of poverty. They have adopted the monthly per capita total expenditure of Rs. 49.09 (rounded to Rs. 49.0) and 56.64 (rounded to rs. 56.6) as the base line poverty estimates corresponding to the requirements of 2100 kilo calories 2400 kilo calories for rural and urban areas respectively at the all India level for the year 1973-74.

(ii) To facilitate comparability of the poverty estimates across states and over time, the national consumption basket corresponding to the calories norms for the base year implicit in these norms, has been used as a standard and has been applied to all the States.

(iii) This standardised national consumption basket has been valued at the prevailing state specific prices to estimate state specific poverty lines. These state specific poverty lines alongwith the state-wise NSS consumption distribution gives the state level proportion of poor.

(iv) At all India level for both rural and urban areas, the proportion of

poor has been derived as a ratio of the aggregate state-wise number of poor to the total all India population for rural and urban areas respectively.

(v) The procedural adjustment of NSS household consumer expenditure distribution to make the NSS national total of household consumption expenditure consistent with the NAS private consumption expenditure has been dispensed away with. This implies that the proportion of poor would be estimated directly from the reported NSS distribution.

(vi) The state specific poverty lines are updated using appropriate state specific cost living indices reflecting the prices relevant to the population around the poverty line.

To arrive at the state specific poverty line corresponding to the base line poverty lines for 1973-74 state-wise price differentials have been estimated separately for the rural and urban areas, using specially constructed cost of living indices. The cost of living indices for the rural areas are based on Consumer Price Index for Agricultural Labour (CPIAL) with 1960-61 as the base. The cost of living indices for urban areas are based on Consumer Price Index for Industrial Workers (CPIIW) with 1960-61 as base. In order to reflect the prices relevant to the consumption basket of the population around the poverty line, the available group-wise CPIAL and CPIIW have been aggregated by using the weights corresponding to the consumption pattern of the middle 40 percent of the rural population and about middle 42 percent of the total urban population by expenditure strata in 1973-74. The resulting composite Consumer Price Index for Middle Rural Population (CPIMR) for 1973-74 for each state and the composite Consumer Price Index for Middle Urban Population (CPIMU) for 1973-74 for each state, both with base 1960-61 have been adjusted for base year price differential across states. Using Fisher's Index, base year price differentials across states had been constructed by Chatterjee and Bhattacharya (1974) for the

rural population for 1963-64. These have been assumed to be same in 1960-61. For urban areas, Minhas et al (1988) estimated state specific price differentials relative to all India for the year 1961-62. These have been assumed to be the same in 1960-61. The price differential across states for 1960-61 alongwith the estimated CPIMR and CPIMU give the price differentials across states relative to all India for rural and urban areas respectively for 1973-74, adjusted for base year price differentials. The vectors so obtained have been applied on the adopted base poverty lines for rural and urban areas to arrive at state specific poverty lines for the year 1973-74. These state specific poverty lines have then been updated for subsequent years i.e. 1977-78, 1983, 1987-88 and 1993-94 using the CPIMR and CPIMU constructed for these years. The state specific pvoerty lines and the respective state-wise unadjusted per capita consumption expenditure distributions as reported by the NSS then yield the respective state-wise proportions of poor. By using the state-wise population figures of the Registrar General of Census for rural and urban areas on the estimated proportions of poor, the total number of poor in each state have been estimated for each of the aforesaid years. These have been aggregated and their ratio to the total rural and urban population in India gives the proportion of poor at all India level. The poverty line and the porportions and number of poor by states have been presented in Tables 2-7.

DATA SOURCE FOR ESTIMATING INCIDENCE OF POVERTY

The data on income distribution by size classes is required for estimation of incidence of poverty. The time series data on income distribution by size classes is, however, not available in India, in the absence of which the country wide data on consumer expenditure, available through NSS consumer expenditure surveys, has become the only source of data for the poverty related studies.

NSSO has conducted household consumer expenditure surveys since its first round started in October, 1950 through the 28th round (1973-74). After 26th round it decided to conduct the survey once in five years only starting from 27th round onwards. So far, five quinquennial surveys have been conducted in 27th (1972-73), 32nd (1977-78), 38th (1983), 43rd (1987-88) and 50th (1993-94) rounds. To maintain the continuity of survey data on consumer expenditure for construction of time series, NSSO has carried out annual thin sample surveys on household consumer expenditure in addition to quinquennial surveys. This has started from 42nd (1986-87) round.

DATA GAPS RELATING TO POVERTY ESTIMATION STUDIES

(1) The poverty line is anchored in a norm for calories consumption which is taken as representing an absolute nutritional requirement based on the age, sex and activity status of the entire population to buy the requisite calories one requires definite income. The measurement of the extent of inequality in the distribution of income in the country, therefore, requires data on household income distribution by size classes. Inspite of its importance and direct relevance to economic policy formulation, the study of income distribution in the country has not come to form a regular exercise mainly because of the paucity of the basic data relating to income distribution. In the absence of time series data on household income distribution, most poverty related studies in India have relied on NSS Consumer expenditure survey data.

(2) Income received by the households by far has the largest share in the total and includes not only the labour and property income generated through production of goods and services but also the transfer incomes received mainly from the government in the forms of relief, unemployment insurance benefits, pension etc. Besides these items of factor income and transfers, households also receive income in kind in the form of community and social services provided by the government, like

education, medical and health and recreation which are received by the households without any financial payments (or concessional payments). Such services are expected to accrue proportionately more to lower income groups than the rest. Comprehensive study of household income distribution should, therefore, include all these aspects.

NSSO have been making attempts to collect data on household income, the latest being through the "Pilot Survey on Income, consumption and Saving" in 1983-84 in both rural and urban areas of five states, namely Maharashtra, Tamil Nadu, Uttar Pradesh, Haryana, Orissa and the metropolitan cities of Calcutta, Bombay, Delhi and Madras. The Primary objective of the survey was to explore the possibility of evolving an operationally feasible and technologically sound methodology for the collection of data on household income through household surveys. In this survey two approaches were adopted for the measurement of household income, namely (i) collection of data on income from different sources of income of sampled households and (ii) collection of data on household consumption and savings which would give an alternative estimate of household income. Unfortunately, it has not been possible so far to evolve a suitable methodology for collection of data on household income through household surveys. The efforts in this direction should, however, be continued.

(3) Non-availability of appropriate state specific price indices that reflects the prices relevant for the population group in and around the poverty line at the state level for a given period and its movement over time is an important gap in the data availability for making state specific estimates of poverty. Steps are required to be taken to construct the price indices representing changes in consumer prices of the poor at relevant disaggregated levels.

(4) Since the NSS distribution of the per capita consumption expenditure is used to derive the estimate of the proportion of poor by

states, it would be desirable for NSSO to conduct the household consumtion expenditure surveys in the north eastern States and the UTs with sufficiently large sample sizes so as to generate robust per capita consumption expenditure distribution in order to get reliable estimates of poverty proportions. At present, however, the sample size for these states/UTs is not large enough to provide reliable estimates of poverty and the recourse is taken to use the per capita consumption expenditure distribution of the neighbouring states to estimate the proportion and number of poor.

(5) The estimate of the incidence of poverty as derived from NSS consumption expenditure distribution provide a composite picture of the number of people whose per capita consumption expenditure is below the desired minimum. It does not, however, provide a complete picture of the state of well being of the population; for example, it does not tell us anything about the living environment. The data is, therefore, required for dissecting the poverty profile in terms of dominant characteristics, namely, their distribution by region, social group, family characteristic, like, size, education age, sex of the head of the household, dependency ratio etc.

REFERENCES

1. Bardhan, P.K. (1971) : On the minimum level of living and the rural poor - A further note. *Indian Eco. Rev.* April 6.

2. Chatterjee, G.S. and Bhattacharya, N. (1974) : Between States Variation in consumer prices and per capita household consumption in rural India. In Srinivasan, T.N. and Bardhan, P.K. (eds.) *Poverty and Income Distribution in India* (1974). Statistical Publishing Society, Calcutta.

3. Dandekar, V.M. and Rath, N. (1971) : *Poverty in Inida*. Indian School of Political Economy, Pune.

4. Minhas, B.S. (1969) : Fourth Plan - Objectives and Policy Frame. Vohra and Co., New Delhi.

5. Minhas, B.S., Jain, L.R. Kansal, S.M. and Saluja, M.R. (1988) : Measurement of General Cost of Living for Urban India — All India and different states. Sarvekshana, Vol. XII, No. 1, July

6. Planning Commission, Government of India (1979) : Report of the Task Force on Projections of Minimum Needs and Effective Consumption Demand.

7. Planning Commission, Government of India (1993) : Report of the Expert Group on Estimation of Proportion and Number of Poor.

8. Rudra, A. (1974) : Minimum level of living. In T.N. Srinivasan and P.K. Bardhan (eds.) A statistical Examination in Poverty and Income Distribution. Statistical Publishing Society, Calcutta.

9. Sukhatme, P.V. (1977) : Measurement of Poverty Based on Nutritional Needs, Bull. Int. Stat. Instt., 47(4), pp 553-556.

10. Sukhatme, P.V. (1978) : Assessment of adequacy of Diets at Different Income Level. EPW, Vol. XIII, pp 1373-84.

Table 2: State Specific Poverty Lines

Sl No	States/UTs	(Rs. per capita per month)									
		Rural					Urban				
		1973-74	1977-78	1983	1987-88	1993-94	1973-74	1977-78	1983-	1987-88	1993-94
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
1.	Andhra Pradesh	41.71	50.88	72.66	91.94	163.01	53.96	69.05	106.43	151.88	278.14
2.	Arunachal Pradesh*	49.82	60.29	98.32	127.44	232.05	50.26	61.38	97.51	126.60	212.42
3.	Assam	49.82	60.29	98.32	127.44	232.05	50.26	61.38	97.51	126.60	212.42
4.	Bihar	57.68	58.93	97.48	120.36	212.16	61.27	67.27	111.80	150.25	238.49
5.	Goa*	50.47	58.07	88.24	115.61	194.94	59.48	73.99	126.47	189.17	328.56
6.	Gujarat	47.10	54.70	83.29	115.00	202.11	62.17	72.39	123.22	173.18	297.22
7.	Haryana	49.95	59.37	88.57	122.90	233.79	52.42	66.94	103.48	143.22	258.23
8.	Himachal Pradesh	49.95	59.37	88.57	122.90	233.79	51.93	66.32	102.26	144.10	253.61
9.	Jammu & Kashmir	46.59	61.53	91.75	124.33	213.83	37.17	55.41	99.62	148.10	248.45
10.	Karnataka	47.24	51.95	83.31	104.46	186.63	58.22	68.85	120.19	171.18	302.89
11.	Kerala	51.68	58.88	99.35	130.61	243.84	62.78	67.05	122.64	163.29	280.54
12.	Madhya Pradesh	50.20	56.26	83.59	107.00	193.10	63.02	74.40	122.82	178.35	317.16
13.	Maharashtra	50.47	58.07	88.24	115.61	194.94	59.48	73.99	126.47	189.17	328.56
14.	Manipur*	49.82	60.29	98.32	127.44	232.05	50.26	61.38	97.51	126.60	212.42
15.	Meghalaya*	49.82	60.29	98.32	127.44	232.05	50.26	61.38	97.51	126.60	212.42
16.	Mezoram*	49.82	60.29	98.32	127.44	232.05	50.26	61.38	97.51	126.60	212.42
17.	Nagaland*	49.82	60.29	98.32	127.44	232.05	50.26	61.38	97.51	126.60	212.42
18.	Orissa	46.87	58.89	106.28	121.42	194.03	59.34	72.41	124.81	165.40	298.22
19.	Punjab	49.95	59.37	88.57	122.90	233.79	51.93	65.70	101.03	144.98	253.61
20.	Rajasthan	50.96	57.54	80.24	117.52	215.89	59.99	72.00	113.55	165.38	280.85
21.	Sikkim*	49.82	60.29	98.32	127.44	232.05	50.26	61.38	97.51	126.60	212.42
22.	Tamil Nadu	45.09	56.62	96.15	118.23	196.53	51.54	67.02	120.30	165.82	296.63
23.	Tripura*	49.82	60.29	98.32	127.44	232.05	50.26	61.38	97.51	126.60	212.42
24.	Uttar Pradesh	48.92	54.21	83.85	114.57	213.01	57.37	69.66	110.23	154.15	258.65
25.	West Bengal	54.49	63.34	105.55	129.21	220.74	54.81	67.50	105.91	149.96	247.53
26.	Delhi	49.95	59.37	88.57	122.90	233.79	67.95	80.17	123.29	176.91	309.48
27.	A & N Island	45.09	56.62	96.15	118.23	196.53	51.54	67.02	120.30	165.82	296.63
28.	Chandigarh*	51.80	66.06	101.35	143.11	242.05	51.93	65.70	101.03	144.98	253.61
29.	Dadra & N. Haveli*	50.47	58.07	88.24	115.61	194.94	59.48	73.99	126.47	189.17	328.56
30.	Lakshdweep*	51.68	58.88	99.35	130.61	243.84	62.78	67.05	122.64	163.29	280.54
31.	Pondicherry*	45.09	56.62	96.15	118.23	196.53	51.54	67.02	120.30	165.82	296.63
	All India	49.00					56.60				
Implicit All India Poverty Line		49.63	56.84	89.45	115.43	205.84					

Note :1. The Poverty lines have been estimated with respect to the 1973-74 base poverty line of Rs. 49.00 and Rs. 56.60 per capita per month for rural and urban areas respectively.

2. Poverty line of Assam is assumed for Sikkim, Arunachal Pradesh, Meghalaya, Mizoram, Manipur, Nagaland and Tripura. Poverty line of Maharashtra for Goa and Dadra & N. Haveli, Poverty line of urban Punjab for Chandigarh, Poverty line of Tamil Nadu for Pondicherry and A & N Islands and kerala for Lakshadweep.

3. Poverty line of Maharashtra and expenditure distribution of Goa is used to estimate poverty ratio of Goa.

Source : Planning Commission, Government of India, New Delhi.

Table 3: Number and percentage of population below Poverty Line by States : 1973-74

Sl No	States/UTs	Rural				Urban	
		No. of persons (Lakhs)	% of persons	No. of persons (Lakhs)	% of persons	No. of persons (Lakhs)	% of persons
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1.	Andhra Pradesh	178.21	48.41	47.48	50.61	225.69	48.86
2.	Arunachal Pradesh	2.57	52.67	0.09	36.92	2.66	51.93
3.	Assam	76.37	52.67	5.46	36.92	81.83	51.21
4.	Bihar	336.52	62.99	34.05	52.96	370.57	61.91
5.	Goa*	3.16	48.85	1.00	37.69	4.16	44.26
6.	Gujarat	94.61	46.35	43.81	52.57	138.42	48.15
7.	Haryana	30.08	34.23	8.24	40.18	38.32	35.36
8.	Himachal Pradesh	9.38	27.42	0.35	13.17	9.73	26.39
9.	Jammu & Kashmir	18.41	45.51	2.07	21.32	20.48	40.83
10.	Karnataka	128.4	55.14	42.27	52.53	170.67	54.47
11.	Kerala	111.36	59.19	24.16	62.74	135.52	59.79
12.	Madhya Pradesh	231.21	62.66	45.09	57.65	276.30	61.78
13.	Maharashtra	210.84	57.71	76.58	43.87	287.42	53.24
14.	Manipur	5.11	52.67	0.75	36.92	5.86	49.96
15.	Meghalaya	4.88	52.67	0.64	36.92	5.52	50.20
16.	Mezoram	1.62	52.67	0.20	36.92	1.82	50.32
17.	Nagaland	2.65	52.67	0.25	36.92	2.90	50.81
18.	Orissa	142.24	67.88	12.23	55.62	154.47	66.18
19.	Punjab	30.47	28.21	10.02	27.96	40.49	28.15
20.	Rajasthan	101.41	44.76	27.10	52.13	128.51	46.14
21.	Sikkim	1.09	52.67	0.10	36.92	1.19	50.86
22.	Tamil Nadu	172.6	57.43	66.92	49.40	239.52	54.94
23.	Tripura	7.88	52.67	0.66	36.92	8.54	51.00
24.	Uttar Pradesh	449.99	56.93	85.74	60.09	535.73	57.07
25.	West Bengal	257.96	73.16	41.34	34.67	299.30	63.43
26.	A & N Island	0.59	57.43	0.15	49.40	0.74	55.56
27.	Chandigarh	0.07	27.96	0.77	27.96	0.84	27.96
28.	Dadra & N. Haveli	0.37	46.85	0.01	37.69	0.38	46.55
29.	Delhi	1.06	24.44	21.78	52.23	22.84	49.61
30.	Lakshdweep	0.18	59.19	0.03	62.74	0.21	59.68
31.	Pondicherry	1.61	57.43	1.13	49.40	2.74	53.82
	All India	2612.90	56.44	600.46	49.01	3213.36	54.88

Note :1. Poverty ratio of Assam is assumed for Sikkim, Arunachal Pradesh, Meghalaya, Mizoram, Manipur, Nagaland and Tripura.

2. Poverty ratio of Tamil Nadu is used for Pondichery and A & N Islands.
3. Poverty ratio of Kerala is used for Lakshadweep.
4. Poverty ratio of Goa is used for Dadra & Nagar Haveli;
5. Urban poverty ratio of Punjab is used for both rural and urban poverty of Chandigarh;
6. Poverty line of Maharashtra and expenditure distribution of Goa is used to estimate poverty ratio of Goa.

Source : *Planning Commission; Government of India, New Delhi.*

Table 4 : Number and percentage of population below Poverty Line by States : 1977-78

Sl No	States/UTs	Rural				Urban	
		No. of persons (Lakhs)	% of persons	No. of persons (Lakhs)	% of persons	No. of persons (Lakhs)	% of persons
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1.	Andhra Pradesh	149.13	38.11	48.41	43.55	197.54	39.31
2.	Arunachal Pradesh	3.26	59.82	0.10	32.71	3.36	58.32
3.	Assam	97.55	59.82	5.83	32.71	103.38	57.15
4.	Bihar	364.48	63.25	37.34	48.76	401.82	61.55
5.	Goa	2.72	37.34	1.16	36.31	3.88	37.23
6.	Gujarat	92.53	41.76	38.35	40.02	130.88	41.23
7.	Haryana	26.43	27.73	9.05	36.57	35.48	29.55
8.	Himachal Pradesh	12.46	33.49	0.58	19.44	13.04	32.45
9.	Jammu & Kashmir	19.04	42.86	2.68	23.71	21.72	38.97
10.	Karnataka	120.39	48.18	47.78	50.36	168.17	48.78
11.	Kerala	102.85	51.48	24.37	55.62	127.22	52.22
12.	Madhya Pradesh	247.98	62.52	54.89	58.66	302.87	61.78
13.	Maharashtra	249.75	63.97	80.16	40.09	329.91	55.88
14.	Manipur	6.09	59.82	0.97	32.71	7.06	53.72
15.	Meghalaya	6.10	59.82	0.69	32.71	6.79	55.19
16.	Mezoram	2.03	59.82	0.28	32.71	2.31	54.38
17.	Nagaland	3.44	59.82	0.30	32.71	3.74	56.04
18.	Orissa	162.50	72.38	13.82	50.92	176.32	70.07
19.	Punjab	18.87	16.37	11.36	27.32	30.23	19.27
20.	Rajasthan	89.66	35.89	27.22	43.53	116.88	37.42
21.	Sikkim	1.41	59.82	0.13	32.71	1.54	55.89
22.	Tamil Nadu	182.50	57.68	72.97	48.69	255.47	54.79
23.	Tripura	9.95	59.82	0.66	32.71	10.61	56.88
24.	Uttar Pradesh	407.41	47.60	96.96	56.23	504.37	49.05
25.	West Bengal	259.69	68.34	50.88	38.20	310.57	60.52
26.	A & N Island	0.71	57.68	0.20	48.69	0.91	55.42
27.	Chandigarh	0.08	27.32	0.95	27.32	1.03	27.32
28.	Dadra & N. Haveli	0.33	37.64	0.16	36.31	0.49	37.20
29.	Delhi	1.35	30.19	16.81	33.51	18.16	33.23
30.	Lakshdweep	0.13	51.48	0.07	55.62	0.20	52.79
31.	Pondicherry	1.65	57.68	1.35	48.69	3.00	53.25
	All India	2642.47	53.07	646.48	45.24	3288.95	51.32

Note :1. Poverty ratio of Assam is assumed for Sikkim, Arunachal Pradesh, Meghalaya, Mizoram, Manipur, Nagaland and Tripura.

2. Poverty ratio of Tamil Nadu is used for Pondichery and A & N Islands.
3. Poverty ratio of Kerala is used for Lakshadweep.
4. Poverty ratio of Goa is used for Dadra & Nagar Haveli;
5. Urban poverty ratio of Punjab is used for both rural and urban poverty of Chandigarh;
6. Poverty line of Maharashtra and expenditure distribution of Goa is used to estimate poverty ratio of Goa.

Source : Planning Commission, Government of India, New Delhi.

Table 5 : Number and percentage of population below Poverty Line by States : 1983

Sl No	States/UTs	Rural				Urban	
		No. of persons (Lakhs)	% of persons	No. of persons (Lakhs)	% of persons	No. of persons (Lakhs)	% of persons
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1.	Andhra Pradesh	114.34	26.53	50.24	36.30	164.58	28.91
2.	Arunachal Pradesh	2.70	42.60	0.12	21.73	2.82	40.88
3.	Assam	73.43	42.60	4.26	21.73	77.69	40.47
4.	Bihar	417.70	64.37	44.35	47.33	462.05	62.22
5.	Goa	1.16	14.81	1.07	27.00	2.23	18.90
6.	Gujarat	72.88	29.80	45.04	39.14	117.92	32.79
7.	Haryana	22.03	20.56	7.57	24.15	29.60	21.37
8.	Himachal Pradesh	7.07	17.00	0.34	9.43	7.41	16.40
9.	Jammu & Kashmir	13.11	26.04	2.49	17.76	15.60	24.24
10.	Karnataka	100.50	36.33	49.31	42.82	149.81	38.24
11.	Kerala	81.62	39.03	25.15	45.68	106.77	40.42
12.	Madhya Pradesh	215.48	48.90	62.49	53.06	277.97	49.78
13.	Maharashtra	193.75	45.23	97.14	40.26	290.89	43.44
14.	Manipur	4.76	42.60	0.89	21.73	5.65	37.02
15.	Meghalaya	5.04	42.60	0.57	21.73	5.62	38.81
16.	Mezoram	1.58	42.60	0.37	21.73	1.96	36.00
17.	Nagaland	3.19	42.60	0.31	21.73	3.50	39.25
18.	Orissa	164.65	67.53	16.66	49.15	181.31	65.29
19.	Punjab	16.79	13.20	11.85	23.79	28.64	16.18
20.	Rajasthan	96.77	33.50	30.06	37.94	126.83	34.46
21.	Sikkim	1.24	42.60	0.10	21.73	1.35	39.71
22.	Tamil Nadu	181.61	53.99	78.46	46.96	260.07	51.66
23.	Tripura	8.35	42.60	0.60	21.73	8.95	40.03
24.	Uttar Pradesh	448.03	46.45	108.71	49.82	556.74	47.07
25.	West Bengal	268.60	63.05	50.09	32.32	318.59	54.85
26.	A & N Island	0.84	53.99	0.26	46.96	1.11	92.13
27.	Chandigarh	0.09	23.79	1.10	23.79	1.19	23.79
28.	Dadra & N. Haveli	0.16	14.81	0.02	27.00	0.18	15.67
29.	Delhi	0.44	7.66	17.95	27.89	18.39	26.22
30.	Lakshdweep	0.09	39.03	0.10	45.68	0.19	42.36
31.	Pondicherry	1.56	53.99	1.72	46.96	3.28	50.06
	All India	2519.57	45.65	709.40	40.79	3328.97	44.48

Note :1. Poverty ratio of Assam is assumed for Sikkim, Arunachal Pradesh, Meghalaya, Mizoram, Manipur, Nagaland and Tripura.

2. Poverty ratio of Tamil Nadu is used for Pondichery and A & N Islands.
3. Poverty ratio of Kerala is used for Lakshadweep.
4. Poverty ratio of Goa is used for Dadra & Nagar Haveli;
5. Urban poverty ratio of Punjab is used for both rural and urban poverty of Chandigarh;
6. Poverty line of Maharashtra and expenditure distribution of Goa is used to estimate poverty ratio of Goa.

Source : *Planning Commission, Government of India, New Delhi.*

Table 6 : Number of percentage of Population below Poverty Line by States - 1987-88

Sl No	States/UTs	Rural				Urban	
		No. of persons (Lakhs)	% of persons	No. of persons (Lakhs)	% of persons	No. of persons (Lakhs)	% of persons
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1.	Andhra Pradesh	96.38	20.92	64.05	40.11	160.43	25.86
2.	Arunachal Pradesh	2.75	39.35	0.08	9.94	2.83	36.22
3.	Assam	73.53	39.35	2.22	9.94	75.75	36.21
4.	Bihar	370.23	52.63	50.70	48.73	420.93	52.13
5.	Goa	1.31	17.64	1.65	35.48	2.96	24.52
6.	Gujarat	74.13	28.67	48.22	37.26	122.36	31.54
7.	Haryana	18.86	16.22	6.51	17.99	25.37	16.64
8.	Himachal Pradesh	7.27	16.28	0.25	6.29	7.52	15.45
9.	Jammu & Kashmir	14.11	25.70	2.85	17.47	16.95	23.82
10.	Karnataka	96.81	32.82	61.80	48.42	158.61	37.53
11.	Kerala	61.64	29.10	26.84	40.33	88.48	31.99
12.	Madhya Pradesh	200.02	41.92	64.29	47.09	264.30	43.7
13.	Maharashtra	186.89	40.78	109.38	39.78	296.27	40.41
14.	Manipur	8.83	39.35	0.46	9.94	5.29	31.35
15.	Meghalaya	5.18	39.35	0.30	9.94	5.48	33.92
16.	Mezoram	1.46	39.35	0.25	9.94	1.70	27.52
17.	Nagaland	3.49	39.35	0.18	9.94	3.56	34.43
18.	Orissa	149.98	57.64	15.95	41.63	165.93	55.58
19.	Punjab	17.09	12.60	8.08	14.67	25.17	13.20
20.	Rajasthan	104.97	33.21	37.93	41.92	142.90	35.50
21.	Sikkim	1.31	39.35	0.04	9.94	1.36	36.06
22.	Tamil Nadu	161.80	45.80	69.27	38.64	231.07	43.39
23.	Tripura	8.49	39.35	0.35	09.94	8.84	35.23
24.	Uttar Pradesh	429.74	41.10	106.79	42.96	536.53	41.46
25.	West Bengal	223.37	48.30	60.24	35.08	283.61	44.72
26.	A & N Island	0.10	1.29	10.15	13.56	10.25	12.41
27.	Chandigarh	0.83	45.80	0.26	38.64	1.09	43.89
28.	Dadra & N. Haveli	0.08	14.67	0.76	14.69	0.84	14.64
29.	Delhi	0.79	67.11	-	-	0.79	67.11
30.	Lakshdweep	0.07	29.10	0.10	40.33	0.17	34.95
31.	Pondicherry	1.33	45.80	1.72	38.54	3.05	41.46
	All India	2318.79	39.09	751.69	38.20	3070.49	58.86

Note :1. Poverty ratio of Assam is assumed for Sikkim, Arunachal Pradesh, Meghalaya, Mizoram, Manipur, Nagaland and Tripura.

2. Poverty ratio of Tamil Nadu is used for Pondichery and A & N Islands.
3. Poverty ratio of Kerala is used for Lakshadweep.
4. Poverty ratio of Goa is used for Dadra & Nagar Haveli;
5. Urban poverty ratio of Punjab is used for both rural and urban poverty of Chandigarh;
6. Poverty line of Maharashtra and expenditure distribution of Goa is used to estimate poverty ratio of Goa.

Source : Planning Commission, Government of India, New Delhi.

Table 7 : Number of Percentage of Population below Poverty Line by States 1993-94.

Sl No	States/UTs	Rural				Urban	
		No. of persons (Lakhs)	% of persons	No. of persons (Lakhs)	% of persons	No. of persons (Lakhs)	% of persons
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1.	Andhra Pradesh	79.49	15.92	74.47	38.33	153.97	22.19
2.	Arunachal Pradesh	3.52	45.01	0.11	7.73	3.73	39.35
3.	Assam	94.33	45.01	2.03	7.73	96.33	40.86
4.	Bihar	450.86	58.21	42.49	34.50	493.35	54.96
5.	Goa	0.38	5.34	1.53	27.03	1.91	14.92
6.	Gujarat	62.16	22.18	43.02	27.89	105.19	24.21
7.	Haryana	36.56	28.02	7.31	16.38	43.88	25.05
8.	Himachal Pradesh	15.14	30.34	0.43	9.18	15.86	28.44
9.	Jammu & Kashmir	19.05	30.34	1.86	9.18	20.92	25.17
10.	Karnataka	95.99	29.88	60.46	.40.14	156.46	33.16
11.	Kerala	55.95	25.76	20.46	24.55	76.41	25.43
12.	Madhya Pradesh	216.19	40.64	82.33	48.38	298.62	42.52
13.	Maharashtra	193.33	37.93	111.90	35.15	305.22	36.86
14.	Manipur	6.33	45.01	0.47	7.73	6.80	33.78
15.	Meghalaya	7.09	45.01	0.29	7.73	7.38	37.92
16.	Mezoram	1.54	45.01	0.30	7.73	1.94	25.56
17.	Nagaland	4.85	45.01	0.20	7.73	5.05	27.92
18.	Orissa	140.90	49.72	19.70	41.64	160.60	48.56
19.	Punjab	17.76	11.95	7.35	11.35	25.11	11.17
20.	Rajasthan	94.68	26.46	33.82	30.49	128.50	27.41
21.	Sikkim	1.81	45.01	0.03	7.73	1.84	41.43
22.	Tamil Nadu	121.70	32.48	80.40	39.77	202.10	35.03
23.	Tripura	11.41	45.01	0.38	07.73	11.79	39.01
24.	Uttar Pradesh	496.17	42.28	108.28	35.39	604.46	40.85
25.	West Bengal	209.90	40.80	44.66	22.41	254.56	35.66
26.	A & N Island	0.73	32.48	0.33	39.17	1.06	34.47
27.	Chandigarh	0.07	11.35	0.73	11.35	0.80	11.35
28.	Dadra & N. Haveli	0.72	51.95	0.06	39.93	0.77	50.84
29.	Daman & Diu	0.03	5.34	0.15	27.03	0.18	15.80
30.	Delhi	0.19	1.90	15.32	16.03	15.51	14.69
31.	Lakshdweep	0.06	25.76	0.08	24.55	0.14	25.04
32.	Pondicherry	0.93	32.48	2.38	39.77	3.31	37.40
	All India	2440.31	37.27	763.37	32.36	3203.68	35.97

Note :1. Poverty ratio of Assam is assumed for Sikkim, Arunachal Pradesh, Meghalaya, Mizoram, Manipur, Nagaland and Tripura.

2. Poverty ratio of Tamil Nadu is used for Pondichery and A & N Islands.
3. Poverty ratio of Kerala is used for Lakshadweep.
4. Poverty ratio of Goa is used for Dadra & Nagar Haveli;
5. Urban poverty ratio of Punjab is used for both rural and urban poverty of Chandigarh;
6. Poverty line of Maharashtra and expenditure distribution of Goa is used to estimate poverty ratio of Goa.

Source : Planning Commission, Government of India, New Delhi.

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट
Gurukula Kangri Vijñāna Patrikā Āryabhaṭa
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AGNIHOTRA-THE AIR PURIFIER

NAVNEET, SUBHASH CHAND AND V. K. SHARMA

संक्षिप्त परिचय: अग्निहोत्र एक विज्ञान है इसमें अग्नि को विशेष साधको से विधिपूर्वक इस तरह प्रज्जवलित किया जाता है कि इसमें ऐसी विशिष्टता बने जो आरोग्य समस्या के समस्त पक्षों पर अपना चमत्कारी प्रभाव दिखा सके। यह वह प्रक्रिया है जो पद्वार्थ को ठोस, द्रव के रूप में न रहने देकर वायुभूत वाष्पीकृत बना देती है। भौतिक प्रयोजनों के लिए अग्निहोत्र और आत्म विकास के लिए यज्ञ का आश्रय लिया जाता है।

प्राचीनकाल से लेकर वर्तमान तक यज्ञ का आत्यधिक महत्व है। यह भारतीय संस्कृति का प्रतीक है, तीर्थो की स्थापना का आधार भी यज्ञ ही थे। यज्ञ में आरोग्य वर्धक, कृमिनाशक एवं रोगविनाशक अदभुत गुण हैं। जहाँ यज्ञ होते हैं वहां सामुहिक रोगों की निश्चित रूप से कमी होती है। जलवायु का परिमार्जन एवं संशोधन होने से उनके रोग कारण दोष हट जाते हैं और आरोग्य वर्धक गुण बढ़ते हैं।

अतः इसी विषय में हमारे द्वारा वनस्पति विज्ञान विभाग में किया गया लघु शोध इस बात की पुष्टि करता है, कि यज्ञ द्वारा वायुमण्डल में सूक्ष्मजीवाणुओं की संख्या में कमी आती है और वायुमण्डल शुद्व हो जाता है।

Agnihotra is the smallest form of Vedic Homa. This sacrificial fire is based on biorhythms of nature. It is a traditional antipollution ritual to be performed daily by a house holder in the morning after sunrise and in the evening before sun set. It is also called Devayajña, according to Maharishi Dayanand Saraswati, the founder of the Arya Samaj, all the methods adopted for keeping the environment clean are regarded as the Devayajña and the Agnihotra. The Agnihotra produces certain specific effects in atmosphere.

Pollution has now become a force that is quickly creating a

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condition of near irreversibility with regard to planetary ecocid. From among the most ancient knowledge known on this planet, the Vedas, comes a solution that was offered and intended to be used in just the circumstances we now face. Vedas consists of miscellaneous kinds of practice . One such practice is being performed in the form of yajña throughout our country.

The seers of Vedic age not only discovered fire, they devised the means of controlling and harnessing it. They finally introduced certain elaborate fire rituals called the Yajñas. The places where these yajñas are performed were know as Yajñashala, they were man's earliest temples of learning, his academics and his open air laboratories. Motivated by the spirit of these yajñas, our seers of yore explored the flora.

The microbes as the causative agents of ailments have been mentioned in Atharveda 1/2/31, 32, 4, 37, 4/23, 29. The microbes ("Krimi") enters the human body through respiration, food and water and cause diseases. These diseases can be cured by putting different parts of medicinal plants having property to destroy microbes into the fire. This clearly indicates that Agnihotra is helpful in destroying micobes present in parts of the enclosed room and even in unapproachable places. The oblations where put into sacred fire, purifies impurities of the air and also penetrate and get diffused in the human body and like a sharp and strong moderator, destory the disease.

Remarkable progress in the spread of Agnihotra has been noted in USA, Chile, Poland and West Germany. In the USA, Agnihotra practitioners are every where, with greatest concentration on the East and West coasts. A fire temple has come up in the Andes Mountains of Chile where Agnihotra is

performed daily at sunset and sunrise and thousands of healings have taken place. West Germany has taken to Agnihotra in a big way. The pioneering work with Agnihotra ash was done and continues to this day. Dozens of external diseases have responded to the ash in preliminary trials. The copper pyramid healing fires were valuable for curing diseased animals as well as humans.

The offering of samidha, ghrita and medicinal plants (Samagri) to the holy fire purifies the air i.e. destroy microbes. Keeping this in view an aeromicrobiological investigation was carried out in the Prarthana Bhavan of Gurukula Kangri University, Hardwar on 28th February, 1997. Aeromicroflora was studied by "gravity petri dish method". The first sampling was conducted one hour before the starting of yajña, the second sampling was done after five minutes of the flame time and the third sampling after half an hour of closing time.

A total of 8 types of fungal spores were trapped from the air during the period of investigation in 3 samplings. The dominant was *Cladosporium cladosporioides* followed by *Penicillium cyclopium* and *Aspergillus niger*. Rest of the fungi were of sporadic occurrence. It was found that *Rhizopus* sp., *Fusarium* sp., *Alternaria alternata* were present only in the 1st sampling and were absent in the subsequent samplings. Moreover, the microbial content of air at a particular sampling time varies. The number of propagules per 100 cm^3 trapped before the Yajña was much higher as compared to the sampling conducted during the Yajña. The microbial load increased in the post Yajña sampling.

The present investigation indicates the reduction in microbial load during the Yajña. This is due to the diffusion of volatile

substances into the atmosphere and thus confirms the antimicrobial effect of medicinal herbs used in havan samagri. Thus Agnihotra is helpful in destroying microbes present in the Prarthana Bhavan and leads to purification of air.

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट
Gurukula Kangri Vijñāna Patrikā Āryabhaṭa
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CLINICAL TRIAL OF TRADITIONAL HERBO MINERAL RECIPE IN LIVER DISEASE AND ALLIED DISORDERS

KAUSHAL KUMAR* AND VINOD UPADHYAY**

Abstract

The study of clinical trial of hepatoprotective activity of traditional herbo-mineral recipe (THMR) has been carried out. Traditional recipe was found to exhibit the most proven overall clinical hepatoprotective effect in nausea and vomiting, anorexia, fever, abdominal distention, oedema, jaundice and hepatic enlargement. The recipe has its origin in the Indian system of 'Folklore' medicine. A search of the literature revealed that all the constituents of THMR viz; Citrus media, Sarjjikhara and Kaparda, have been acclaimed to have beneficial effects on various liver ailments. The drug was accected by patients of all age groups and sexes with out any toxicity. There is a wide scope of such drug for global implication, because no suitable drugs for liver disorders are available in the modern systems of medicine. Further study of isolated herbo-mineral chemicals in relation to their potential antiviral activity against hepatitis viruses has been suggested.

Introduction

The liver plays a vital role in the metabolic activity of the human body. It has numerous functions, e.g. production and secretion of bile, maintenance of blood sugar level, regulation of protein and fat metabolism,

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formation of ketone bodies and plasma proteins, detoxication, erythropoiesis etc. (1). It is obvious from the nature of the diverse functions that the diseases of the liver are bound to have an adverse effect on the entire system.

In developing countries malnutrition contaminated food and water resulting in amoebiasis, unsanitary environment and unhygienic manner of living due to illiteracy, consumption of alcolol and excessive doses of strong antibiotics are some of the main causes of liver ailments. These ailments include Jaundice, Hepatitis, Cirrhosis of liver; which results into anorexia, mild fever, nausea and vomiting, oedema on foot, pruritus, anaemia and constipation. Since our resources are somewhat limited, we need indigenous medicines free from undesirable side effect and simpler technologies than that being utilized by the multi-national drug industry to serve our needs in the field of human health-care. An effective medicine prepared from natural sources containing salts of Na, K, Ca, etc. being rapidly absorbed from the alimentary canal (2), should therefore find a high level of accecptance by the patients suffering from liver diseases.

We describe herein clinical trial of traditional herbo-mineral recipe (THMR), with three ingredients viz., Citrus media, Sarjjikhara and Kaparda, on fifty patients. The physico-chemical study of THMR has been conducted and reported in our previous papaer (2). Citrus media juice is considered refrigerent, appetizer, antiseptic, Stomachic antiscorbutic (3). The ingredients mentioned above are in use from the ancient time (Charaka) and have been occurring in some of Ayurvedic formulations like 'Cankha Bati'. The pills of 'Cankha Bati' reported to cure loss of appetite, indigestion, abdominal dropsy, sour bile. It enkindles the digestive fire (4).

Materials and Methods

In the present study, fifity cases were selected from the indoor and outpatients departments of state Ayurvedic College/ Hospital. The hospital caters to the medical need of 40 nearly villages besides Hardwar, Jwalapur, Kankhal and BHEL area. The patients selected were rom diverse income groups (Table-I). Deficient diet resulting in malnutrition was the chief factor in leading to liver disorders in the Ist and IInd groups while rich diet over eating and more than moderate consumption of alcohol were responsible in the 3rd group for liver diseasas.

TABLE-I

Level of Income	No. of patients
Low	23
Middle	22
High	05
Total	50

A breakdown of the patients by age and sex is given in table II.

TABLE-II

	Age Years			
Sex	1 to10	11to35	36to60	Total
Male	05	20	05	30
Female	04	13	03	20
Total				50

Diagnosis of the patients with liver trouble and the prevalent symptoms are tabulated below (Table-III) by the number of patients manifesting the varied symptoms.

TABLE-III

Prevalent signs and symptoms	No. of patients
Nausea and Vomiting	45
Anorexia	49
Fever	35
Abdominal Distention	20
Oedema	25
Jaundice	15
Hepatic Enlargement (1 to 4 fingers)	35

Ingredients of THMR

Each ml of recipe consists of the following ingredients.

Citrus media	500 mg
Sarjjikhara	16 mg
Kaparda	18 mg

Does : 1/4 Teaspoonful in infants and children up to 10 year, thrice-times a day, 1/2 Teaspoonful to 1 teaspoonful in adults thrice times a day. The recipe was administered with or without hot water depending upon the patients response to the limey-taste of the drug.

Dietary Restriction: Fat free diet was recommended.

Duration: The THMR was administered every day for 3 to 4 weeks.

Results and Discussion

Administration of THMR caused the cessation of nausea and vomiting in all cases with the very first dose of the medicine. Beneficial effects of THMR correlated with the duration of treatment on various symptoms of the hepatic disorder in table-IV.

TABLE-IV

Prevalent signs & Symptoms	Duration of Treatment	Result
Nausea &Vomiting	Ist day	In most of the cases complete cure.
Anorexia, Fever & Abdominal Distention	Ist week	Fever abated, abdominal distention disappeared patients had a feeling of hunger.
Jaundice	3 to 7 days	Paleness of face, conjunctiva & other related symptoms disappeared.
Oedema	2nd & 3rd week	Abated.
Hepatic enlargement	4th week	In most cases1/2 cm. redution in the enlarged liver size every 3rd day.

During the first week of treatment patients with showed significant improvement and abatement of fever and abodominal distention too, was also observed. Patients with jaundice showed very significant recovery from 3 to 7 days, paleness of face, conjunctiva and other related symptoms viz; pale colour of urine were disappeared during this period. Oedema was abated during the 2nd ane 3rd week of the administration of THMR. Patients with hepatic enlagement were clearly and significantly benefitted. Of 35 cases of hepatic enlargement 31 (i.e.88.5%) showed an averge 1/2cm. redution of enlarged liver size every 3rd day. Overall, considertion of the various symptoms, the time period of treatment and the benefit derived therefrom are tabulated in Table-V.

As is appearent from Table-V, traditional herbo-minaral recipe (THMR) produced a very satisfactory response in 70% of the patients

suffering from heapatic disorders whereas about 26% showed moderate to good improvement. A searh of the literature revealed that the various constituents of THMR individually have been acclaimed to have beneficial effects on various hepatic disorders. Thus Citrus media has been cited by Daljit Singh (5) and Nadkarni (6) as a medicine for liver diseases. Efficacy of Sarijjikhara in treatment of malfunction of the liver has been documented in Indian Meteria medica (7) and by Chopra (8). Kaparda has been mentioned to cause significant improvement in liver problem by Daljeet Singh (5) Nadkarni (6) and Chopra (8).

TABLE-V

No.of Patients	%of Patients in the study	Duration of Treatment	Results
35	70	1-2 weeks	Very good improvemennt.
10	20	2-3 weeks	Good improvement.
3	6	3-4 weeks	Moderate improvement.
2	4	4 weeks	No response.

The traditional herbo-mineral recipe (THMR), which includes all above three ingredients thus creates no surprise as to its efficacy in treatment of hepatic disoders. Moreover, during the study it was observerd that the medicine was accepted by patients of all age groups and sexes without any toxicity. This is an important factor since some of the available medicines have bitter herbal glycosides not easily accepted by some patients. As table V shows, approximately 5% of patients showed no response to THMR. It should be mentioned here that longer treatment in the improved patients may lead to almost complete recovery from existing hepatic disorders.

Conclusion

In conclusion, we believe that the results obtained in this study that the wide acceptability, promptness in eliciting response, easy adiministration and high percentage of satisfactory improvement by THMR shold recommend its use to the patients suffering from hepatic disorders. In this regard this recipe has advantage of using substances with which the system is already familiar. There is a wide scope of such preparation for global implications in hepatic disorders. Traditional recipe (THMR) may provide a new impetus to researchers and help to find new products to combat deadly hepatic viruses responsible to set untimely death. Further study of isolated herbo-mineral chemicals in relation of their potential antiviral activity against different hepatitis viruses has been suggested.

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References

1. William Boyd, A Text Book Pathology, Structure and Function in disease, Publisher Lea & Febiger, Philadejphia, 1970, pp 859.
2. Kaushal Kumar, V. P. Upadhyay, Chemistry of Traditional recipe in Hepatic disorders, J. of Eastern Institute 'TOHO' vol. 10, 1994, Tokyo-Japan pp 136-144.
3. R.N. Chopra, Glossarry of Indian Medical Plants, 1992 CSIR, New Delhi.
4. N. N. Sen Gupta, The Ayurvedic system of medicine vol. II, 1984, Neerj Publishing House, Delhi.
5. Daljeet Singh, Unani dravyayadarsh II; Ayurved and Tibbi Academy, Lucknow, 1974,pp-522.

6. A. K. Nadkarni, Indian Materia Medica I; Popular Prakashan Bombay, 19976, pp-348.

7. A. K. Nadkarni, Ibld vol. I, pp-101.

8. R. N. Chopra, Indigenous drugs of India, U.N. Dhar & Sons, Calcutta, 1958.

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट
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OZONE DEPLETION - AN ENVIRONMENTAL CHALLENGE

A. K. CHOPRA *

सारांश

प्रदूषण की समस्या विश्वव्यापी है। आधुनियक युग में हो रहे औद्योगिक विकास ने वायुमंडल में विभिन्न हानिकारक गैसों मुख्यतः नाइट्रोजन, क्लोरो-फ्लोरो-कार्बनस की मात्रा इतनी अधिक बढ़ा दी है कि वायुमंडल के ऊपरी क्षेत्र समताप मंडल में प्रयुक्त मोटी ओज़ोन परत के क्षीण होने का कारण बन गई है। ओजोन परत का समताप मंडल में समरूप बने रहना बहुत ही महत्वपूर्ण है। यह परत सूर्य-किरणों में उपस्थित पराबैंगनी 'बी' किरणों के भू-मंडल में प्रवेश को रोकती है। पराबैंगनी किरणें भू-मंडल में उपस्थित जीव-जन्तुओं और पौधों के लिए ही नहीं बल्कि मानव-जाति के लिए भी अत्यन्त हानिकारक हैं जिसके भयावह दूरगामी परिणाम हो सकते हैं। ओजोन की इस परत को सुरक्षित रखना मानव के लिए चुनौती बन गया है। यद्यपि इस परत को क्षीण होने से बचाने के लिए विश्व स्तर पर प्रयास जारी हैं, परंतु यह तो आने वाला समय ही बता पायेगा कि यह प्रयास कहाँ तक कारगर सिद्ध हुए हैं।

The environmental challenges facing humanity today are more disastrous now than at any time in our history. The scale and magnitude of environmental degradation is so high that the climatic pattern has changed so much that many species of plants, animals and organisms are at the verge of extinction. If this degradation is kept unchecked, life on earth would become impossible. Today, one of the main challenges to earth's environment is Ozone layer depletion.

What is Ozone layer ?

Ozone is present at all altitudes in the atmosphere, mainly in the

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upper atmosphere i.e. stratosphere extending from 10 km to 40 km depending on latitude. This upper layer of atmosphere having ozone is known as ozone layer. The stratosphere has a low temperature, virtually no clouds, dust or water vapour. The ozone layer in stratosphere is of great significance as it acts as a protective filter by absorbing ultraviolet (UV) radiations from sun and thus prevents the earth from its harmful effects.

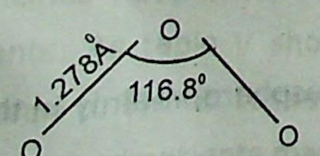
Ultra-violet rays

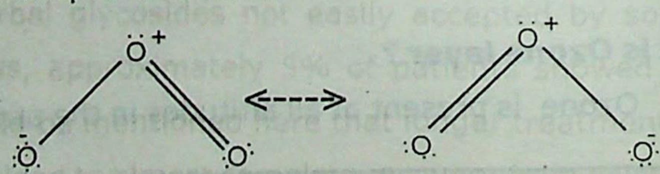
Sun emits radiations of all the wavelengths- from very small gamma rays to very large radiowaves. Of these, the radiations of wavelengths shorter than 295 nm are dangerous but do not reach the earth's surface. They are absorbed by different atmospheric gases. The rays of wavelengths, greater than 200 nm are considered UV rays. On the basis of wave-lengths, UV rays are divided into three categories : i) UV-A rays (320-400 nm) ii) UV-B rays (280-320 nm) and iii) UV-C rays (200-280 nm).

UV-A rays are not affected by ozone. UV-B rays are harmful among all UV rays. These are absorbed only by ozone and are of great concern. Any loss or destruction of startospheric ozone layer could mean greater amount of UV-B rays reaching the earth. UV-C rays are absorbed by other atmospheric constituents besides ozone.

What is Ozone ?

Ozone (O_3) is a deep blue gas constituted of chemically bonded oxygen atoms. It has a v-shaped structure.





i) *Ozone molecule (Geometry)* (ii) *Resonance hybrid of Ozone*

How Ozone was Formed ?

The evolution of life on earth indicates that initially the atmosphere was devoid of oxygen but photosynthetic activities of blue-green algae and other anaerobic unicellular organisms helped in building up oxygen level in atmosphere. It is believed that first multicellular organisms appeared when oxygen level increased by 3% than that of the than present value. The concentration of oxygen increased gradually afterwards with the evolution of more complex multi-cellular organisms. The building up of oxygen level in atmosphere was found to be very beneficial because it helped in retaining the deadly UV radiation of sun in upper atmosphere only and thus was formation of ozone layer in stratosphere.

Concentration of Ozone

The Ozone concentration remains about 10 ppm in the stratosphere whereas its concentration is only 0.05 ppm in troposphere. The concentration is negligible as compared to other gases i.e. parts per million of total gas density. Its peak concentration has been observed to be around an altitude of 30 km in the tropics and around 15-20 km over the polar regions. The ozone found over the tropics is distributed poleward through the stratospheric circulation, particularly in the upper atmosphere where the airflow is the strongest and most meridional. Since the level of peak ozone is considerable higher in altitude in the tropics, ozone descends as it moves toward the poles, where, because of very low photochemical destruction, it accumulates, particularly in the winter hemisphere. Some ozone eventually enters the troposphere over the poles. Seasonal variations are much stronger in the polar regions, reaching 50% of the annual mean in the Arctic. The average pattern in ozone concentration varies not only from season to season but on day to day basis too as the dynamic circulation of the stratosphere overwhelms photo - chemical influences.

Equilibrium in Ozone concentration

The nature maintains an equilibrium in ozone concentration. In stratosphere, atmospheric oxygen absorbs UV radiation 240 nm and photo-dissociates into two oxygen atoms. These atoms subsequently combine with molecular oxygen of upper stratosphere, producing ozone.

$$0 \quad + \quad 0 \quad \longrightarrow \quad 0_2 \text{ (Oxygen)}$$
$$0_2 \quad + \quad 0 \quad \longrightarrow \quad 0_3 \text{ (Ozone)}$$

Ozone also absorbs short wavelength UV radiations, releasing oxygen atoms.

$$0_3 \quad \longrightarrow \quad 0_2 \quad + \quad 0 \text{ (Odd oxygen atom)}$$
$$20_3 \quad \longrightarrow \quad 30_2$$

This natural mechanism does not upset the ozone equilibrium because of the fact that loss of ozone by natural process is compensated by creation of ozone through atmospheric circulation. But this natural equilibrium is being upset with the introduction of anthropogenic gases in the atmosphere.

Sources of pollutants

Supersonic air-crafts emit a fairly large quantity of pollutants like water-vapour and oxides of nitrogen. Nuclear explosions produce large quantities of oxides of nitrogen. Volcanic eruptions and chimneys of several factories emit sulphate aerosols. Agricultural activities result in increase of methane concentration. Vehicles, industrial chimneys, smelting furnaces emit carbon-dioxide and other gases. Air-conditioning, refrigeration, aerosols, electric and metal cleaning, foam-insultation, fast-food containers and solvents used for

cleaning of microchips and fire-extinguishers emit a wide range of chloro-flouro-carbons (CFCs).

How the Ozone layer is depleted ?

Anthropogenic activities result in the increase in concentration of important gases of hydrogen species (OH and CH), nitrogen species (N) and chloro-flouro-carbons (Cl) species in the troposphere. These gases then escape into stratosphere by normal atmospheric circulation. These pollutants intercept free O and form different oxides like NO, N_2O, NO_2, Cl0 etc. and thus a balance of ozone and oxygen is disturbed. The gases not only react directly with ozone or free oxygen atoms but also may combine in several ways in chain processes to deplete ozone concentration.

Influence of Hydrogen species

The hydroxyl radical (OH) has a short life-time in both the troposphere and stratosphere. In the upper stratosphere, where photochemical reactions are strong, OH removes odd oxygen by forming odd hydrogen. The following reactions exist.

$$H_20 \longrightarrow OH + H$$
$$OH + O \longrightarrow H + O_2$$
$$H + O_3 \longrightarrow OH + O_2$$
$$H + O_2 + M \longrightarrow HO_2 + M \text{ (Third body)}$$
$$OH + 0 \longrightarrow OH + O_2$$

Methane also contains hydrogen. This hydrogen reacts with odd chlorine atoms which remove chloride from the active phase and places it in the reservoir status as hydrochloric acid.

$$H + Cl \longrightarrow HCl$$

Influence of Nitrogen species

In the stratosphere, the processes controlling the ozone balance are dominated by NO and NO_2. The major source of NO in the stratosphere is N_2O from the troposphere. The following reactions exist :

$$N_20 + O \longrightarrow 2NO \quad \text{(Photodissociation)}$$
$$NO + O_3 \longrightarrow NO_2+O_2 \quad \text{(Destruction)}$$
$$NO_2 + O \longrightarrow NO+O_2 \quad \text{(Dissociation)}$$
$$NO_2 \longrightarrow NO+O \quad \text{(Photodissociation)}$$

Influence of Chlorine Species

CFCs are not soluble in water and thus are not washed out of the troposphere and therefore, do not come back to earth with precipitation. The photochemical reactions are effective in stratosphere. UV rays break CFCs into their constituents of carbon, fluorine and chlorine. Chlorine is highly reactive and attacks the ozone in a chain reaction which results in thousands of ozone molecules to break into oxygen. The following reactions exist :

Direct reactions -

$$Cl + O_3 \longrightarrow ClO + O_2 \quad \text{(Destruction)}$$
$$ClO + O \longrightarrow Cl + O_2 \quad \text{(Release of Cl)}$$
$$ClO \longrightarrow Cl + O \quad \text{(Photodissociation)}$$

Indirect reactions -

$$Cl0 + NO \longrightarrow Cl + NO_2$$

It has been seen that chloride alone is so reactive that it can destroy several thousands of ozone molecules. Thus, influence of CFCs are more harmful than H, NO and CH.

Antarctic Ozone hole

Scientists have produced graphic colour-enhanced photo-graphs of the depletion of ozone over Antarctica with the help of Total Ozone Measuring System (TOMS) on the Nimbus satellite. This has been reported as ozone hole. The hole represents a depletion of ozone concentration over Antractica, not an empty space in the atmosphere. Average ozone cocentration within the Antractic ozone hole for the latitudinal band 70-80^0 south has been observed to be as follows :

Year	Zonal mean Ozone (DU)
1970	306
1971	334
1983	245
1984	240
1994	90

Since last three decades, ozone concentration has decreased considerably of the normal thickness. The surface area of the hole has been estimated to be about 23 million sq. km. Every 1% drop in ozone means about 1.3 to 1.5% more UV radiation penetrating the stratosphere.

Is three a hole over Arctica ?

In Arctic region, ozone concentration has also decreased but in comparison to Antarctica, it is very low. It is observed that if hole develops in arctic region, its consequences would be very dangerous.

Impact of Ozone Depletion

There are several potentially serious problems that may occur from increased UV-B radiation due to depletion of ozone concentration. i) UV-B radiation increases the chances of surface skin cancer

known as Melanoma. It is malignant and can cause death. Melanoma occurrence is on the increase, probably due to a variety of factors associated with the life-style of people. There is as yet no direct evidence to link melanoma increases to changes in the stratospheric ozone layer. ii) Two other types of cancer may also develop such as Basal-cell carcinomas and squamous-cell carcinomas. Though they may not cause death, they may be responsible for disfiguration of humans.iii) Increase in the incidence of cataracts and interference with human immunity system are other possible influences. iv) A more serious long-term threat is damage to cell DNA and genetic structure in not only humans but in plants, animals and organisms. v) UV-B radiation could also damage many crops and aquatic organisms, including shrimps and eggs of fishes. vi) The reproductive process of plants and animals may also be affected.

Who is responsible ?

USA comprises only 5% of world population but it uses 35% of the raw materials of the World. More than 40% of carbon-dioxide and other green-house gases thrown into the atmosphere comes from USA and Europe. They release 70% of CFCs every year where as India and China put together having maximum population release only 1.8% of CFCs. Thus USA and Europe are mainly responsible for ozone depletion.

Remedies

The remarkable step in preventing ozone depletion would be to limit the emission of CFCs, the main agents of ozone destruction. A delay in severely limiting CFCs production and use now could have serious consequences for life on earth over the next few decades. However, a ban on CFCs use would interfere with life styles in many devloped countries and with the attempts of developing countries to modernise industry and raise living standards. Several National and

International conferences have been convened in different parts of the world. The signing of Montreal protocol 1987 by 24 nations including USA and Britain; and its modification in 1990 is a historic step in halting CFCs altogether by 2000 AD. They have evolved appropriate ozone-friendly substitutes and can therefore afford to hasten the phasing out of CFCs earlier than the deadline. Europe, Germany and Scandinavian nations have also achieved 100% phase out. In South-Africa 85% cut has been recorded.

Many scientists of USA are finding ways to plug the ozone holes. They are injecting the alkanes, ethanes or propanes into Antarctic atmosphere. The alkanes would rapidly react with the ozone-destroying chlorine atoms thus immobilizing them. The alkanes can be delivered to an altitude of 15 km with a fleet of several hundred large air planes. Inspite of these efforts, the hole is increasing in expanse, because 20 million tonnes of CFCs having a life-span of 60-100 years, have reached into the atmosphere since last fifty years. It is expected that CFCs would reach their peak by the end of the next decade. The recovery phase will begin with the hole shrinking gradually and disappearing completely by next fifty years or so. Only time will tell as to how much success has been achieved in compensating the loss already made, and thus saving the humanity from the consequences of ozone depletion.

SUGGESTED READINGS

1. Bridgman, Howard - *Global Air Polution : Problems for the 1990s*. CBS Publishers & Distributors (P) Ltd., New Delhi, 1992
2. Howard, E.H. - *Air Pollution Control,* Ann Arbor Science Publishers, Ann Arbor MI, 1979
3. Iqbal, S.A. and Mido, Y. - *Chemistry of Air and Air Pollution*, New Delhi, 1995

4. Sharma, B.K. and Kaur, H.-*Environmental Chemistry*, Goel Publishing House, Meerut, 1997.

5. Stern, A.C. - *Air Pollution*, 3rd Ed., Academic Press, New York, 1977

6. Turk, J. and Turk, A. - *Environmental Science*, 2nd Ed., Saunders College Publishing, New York, 1984

"Woodman spare the tree,
touch not a single bough,
In past, it sheltered me,
and I will protect it now"

G.P.Morris

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट
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LIFE AND WORKS OF ŚRĪPATI

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Abstract

In this paper we discuss Mathematics of *Vedic* tradition and give an account of life and works of *Śrīpati*.

"*Vedās* are the sources of eternal knowledge".

The meaning of the *Vedic* Mathematics is Mathematics as found in the *Vedās* (or *Vedic* Literatures) or in the *Vedic* times. More widely interpreted it can mean Mathematics as developed in *Vedic* tradition.

Vedic Mathematics is supposed to be earlier than the ancient Mathematics of Babylonia, China and Egypt see [2].

According to Seidenberg the ritual origin of all Ancient Mathematics and accepting the diffusion theory of great ideas, a new hypothesis was emerged out. B.L. van der Waerden views that "Mathematics was invented by the Indo-Europeans before their dispersal between about 3500 B.C. and 2500 B.C."

Ṛgveda, Yajur-veda, Sāma-Veda, Atharva-veda, Saṁhitas (*namely, ṚK, Sāma, Yajus* and *Atharvan*), *Brāhmaṇans, Āraṇyakas, Upaniṣads*, Six *Vedāngas* (namely, *Śikhṣa, Kalpa, vyākaraṇa, nirukta, chanda* and *jyautiṣa*), *Upavedas* (namely, *Āyurveda, Dhanur-veda, Gāndharva-veda* and *Sthāpatya-veda)*, the *Prātiśākhyas, Baudhayānaśulbasūtra, Āpastambhaśulbasūtra, Kātyāyanaśulbasūtra, Mānavaśulbasūtra*,

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Maitrāyanaśulbasūtra, Varāhaśulbasūtra, Hiraṇyakeiśulbasūtra and *Pāniṇisūtra* are the various famous ramarkable sources of *Vedic* Mathematics.

Āryabhaṭa I (b.476 A.D.), *Brahmagupta* (b.598 A.D.), *Bhāskarā* I (c.629 A.D.), *Śrīdhara* (c. 750 A.D.), *Mahāvīrācārya* (c.850 A.D.), *Āryabhaṭa* II (c.950 A.D.), *Śrīpati* (1039 A.D.), and *Nārāyaṇa Paṇḍit* (c.1356 A.D.) are the famous ancient Indian mathematicians.

Śrīpati was most prominent Indian mathematician, born in *śaka* 921 (999 A.D.) at *'Rohinīkhaṇda'*. He was son of *Nāgadevbhaṭṭa* and grandson of *Keśavabhaṭṭa*. *Śrīpati* was *Maheśvara* or *Śaiva* by religion and a *Kāśyapa* by lineage. He composed his mathematical and astronomical treatises in *Rohinīkhaṇda* as it is clear from the following verses :

(I) 'भट्टकेशवपुत्रस्य नागदेवस्य नन्दनः ।
श्रीपती रोहिणीखण्डे ज्योतिः शास्त्रमिदं व्यधात् ।।'

(II) " 'कश्यप' वंशपुण्डरीकखण्डमार्तण्डः केशवस्य
पौत्रः नागदेवस्य सूनुः श्रीपतिः संहितार्थमाभिधातुमिच्छुराह' "

According to Prof. K. N. Sinha see [4, p.112] that the *Rohinikhanda* is located in the *Buldhana* District of *Maharastra*. His period of activity was from (1039 A.D.-1056 A.D.).

It appears that after studying, *Bakṣālī Manuscript* (200 A.D.), *Āryabhaṭa* I (b. 476 A.D.)., *Varāhamihiar* (505 A.D.), *Brahmagupta* (b. 598 A.D.), *Bhāskarā* I (c.629 A.D.), Lalla (c.748 A.D.), *Śrīdhara* (c. 750 A.D.), *Govindasvāmī* (800-850 A.D.), *Skandasena* (c.9th century beginning), *Mahāvīrācārya* (c.850 A.D.), *Prthudakasvāmī* (860 A.D.), *Āryabhaṭa* II (c.950 A.D.) and etc., *Śrīpati* cristised the unique doctrines of the Jains about their views two suns, two moons, a double set of stars and planets and the pyramidal shape of Meru (mountain) see [3, p.63].

Śrīpati was not only a renowned mathematician but also a famous astronomer and astrologer. He composed the following books of Mathematics and Astronomy without any imperial support : (i) *Śrīpati-Paddhati,* (ii) *Śrīpati-Ratnamālā*, (iii) *Śrīpati-Nibandha,* (iv) *Śrīpati-Samuceaya*, (v) *Daivajnāvallabha*, (vi) *Bījagaṇita*, (vii) *Ratnasāra*, (viii) *Gaṇitatilaka*, (ix) *Dhikoṭidakarana*, (x) *Siddhānta-Sekhara*, (xi) *Dhruvāmāsa Karaṇa*.

The *Gaṇitatilaka* is of the best work of *Pātī-gaṇita*. In the beginning of *Gaṇitatilaka, Śrīpati* first offers his salutations to God as it is clear from the following verse :

(I) रूपोज्झितं रूपयंतं स्वरूप-मात्मस्वरूपं परमं प्रणम्य।
करोमि लोकव्यवहारेहेतो-र्विचित्रवृत्तां गाणितस्य पाटीम्।।

which means see [4, p.114]

"Having bowed to the Supreme Who is forsaken as well as possessed of form, Who is his own nature, (and also) the nature of the soul itself, I prepare this Arithmetic, having variegated verses, for public use".

In ancient India, *Gaṇitatilaka* (a book of Pātī-Mathematics) was being taught in a traditional way for the mental and logical development of the students. *Gaṇitatilaka* is entirely composed in Sanskrit verses in various meters, comprises 125 verses with some hemi stich. The commentor *Simhatilaka Sūri* of *Gaṇitatilaka* (see [1], [3] & [4]) discusses, 18 notational places (numeration by tens upto 10^{17}), the terminology for various units of measurement, 8 sort of fundamental operations (addition, subtraction, multiplication, division, square, square-root, cube and cube-root of whole numbers), squaring of fractions, division of fractions, reduction of five standared classes of fractions (*bhāga-jāti, prabhāga-jāti, bhāganubandhu, bhāgapavāha* and *vallīsavarṇana*), problems related to the solution of three and inverse rule of three, rule of

five and dependent rules of barter and sell of living animals, simple interest, usvry, rule conversion of a number of bonds into a single one and rule equating shares of bonds into a single one and rule for equating shares of capital given by different persons for different periods of time.

Śrīpati has given the following method for extracting cube root, which is given by the following slokes:

(I) घनोऽघनद्वन्द्वमिति प्रपात्य
घनं घनानन्मूलमघः पदस्य।
नयेत् तृतीयस्य हरेच्च शेषं
त्रिनिघ्नकृत्यास्य निजोज्य लब्धम्।।
पङ्क्त्यां ततस्तत्कृतिमन्त्यनिघ्नीं
त्रिसङ्गुणां चापनयेंद् घनं च।
विधानमेतद् गणकेन नूनं
पुनर्विधेयं घनमूललब्ध्यै।।

Which mesns (see [4, p. 118])

The places starting from the units place are to be divided. Thus : one 'cubic place' and than a pair of non-cubic' places. Having subtracted the greatest possible cube from the last 'ghana' place, one should bring the cube-root under the third place of the resuting number and divide out the remainder, i. e. the number to the left of the cube-root by thrice its square. Having placed the quotent in the line of the root, one should then subtract the square of this as multiplied by thrice the last numbers, i.e. the number to the left of the quotient in the line of the root, from the number above. One should also subtract the cube of the quotient from the cubic place above. This operation is surely to be performed by ganak repeatedly, for the obtainment of cube-root.

This method ia illustrated by the following examples.

Example 1: Compute the cube root of 31855013.

First we divide the given number into cubic and non cubic places. The digit at the unit place is a cubic digit and the next two digits are non cubic digits. The rest of the digits in the number can be arranged in the similar way as $\overset{\delta\sigma\delta\delta\sigma\delta\delta\sigma}{31855013}$.

Here the Sigma (σ) represents cubic and Delta (δ) non cubic places.

It can be seen that the digit at the last cubic place is 1 and the number up to this last place is 31. We observe that the greatest possible cube less than or equal to 31 is 27. So, we claim that $a^3 = 27$, i.e.,$a=3$. Subtracting 27 from 31 we get 4. Now, we take remaining digit of the number one by one from left to right. Thus putting one digit namely, 8 after 4 we make it 48. Next, we divide 48 by $3a^2$, i,e., 27. The quotient obtained by this division is b (=1) and the remainder is 21. We take that next digit from the number and make the 21 as 215. From this 215, we subtract $3ab^2$, i.e., 9 and obtain 206. We again take the next digit from the number and make 206 as 2065. Now, we subtract b^3 (=1) from 2065 and obtain 2064. In the next operation, we again take the next digit from thee number, i.e., O and divide 20640 by $3\,(10a+b)^2=2883$. The quotient, thus obtained is c which is equal to 7 here and the remainder comes out to be 459. Taking the next digit of the number, we have 4591. Now $3(10a+b)\,c^2$, i.e., 4557 is subtracted from this number which gives us 34. Again, taking the next digit, i.e., 3, 34 becomes 343. Next, we subtract c^3 (=343) from this number and thus difference is zero and the procedure terminate. We have the cube root of 31855013 from this process as $100a+10b+c=317$.

REFERENCES

(1) B. Datta and A. N. Singh, History of Hindu Mathematics, (Part I&II), Asia Publishing House, Bombay 1962.

(2) R.C. Gupta, Six Type of *Vedic* Mathematics, *Gaṇita Bhāratī*, Vol. 16, 1-4 (1994), 5-15.

(3) H. R. Kapadia (ED.), *Gaṇitatilaka of Śrīpati*, Gaekwad's Oriential Series, Baroda, 1937.

(4) K. N. Sinha, *Śrīpati Gaṇitatilakas*: English Translation, *Gaṇit-Bhāratī*, 1982, 112-133.

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट
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'CALCULUS-ITS USE AND DEVELOPMENT IN ANCIENT INDIA'

PARMESHWAR JHA

Abstract

The origin of calculus in India can be traced back to the period of *Vedas*. The *Atharva Veda pariśiṣṭa* contains a number of *Sūtras* which deal with application and uses of principles of differential as well as integral calculus. Later on, methods of obtaining differentiation of some trigenometrical functions have been employed and the process of integration for finding the surface and volume of a sphere has been introduced. The credit must be given to *Bhāskara* II for having introduced the mean value theorem of differential calculus and also for having used the method of integration.

The most remarkable mathematical achievement of the 17th century is the invention of calculus by newton (1664) and Leibnitz (1676). It was created to treat major scientific problems relating to instantaneous motion, tangent to a curve, max. or min. value of a function, lengths of curves, areas, volumes, centre of gravity of bodies, etc. These problems were solved even earlier than the 17th century but methods were not as scientific as the modern one is. A crude integral calculus was already in use and some approach had been made in the process of differential calculus. Some of the ancient Greeks used and make the application of 'methods of exhaution' for solving problems relating to area, volume, etc. There are grounds to believe that applications and uses of some of the basic principles of calculus were in vogue in ancient India, too. Several

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attempts have been made in the past to take stock of outstanding contributions of India to different branches of mathematics, but very little efforts have been made so far to elucidate its achievements in the field of calculus[1]. The present paper is an attempt in this direction.

The origin of calculus in Indian can be traced back to the period of *Vedas* (3000 B.C.), the oldest religious scriptures of the whole world. Scientific ideas alongwith religious and metaphysical thoughts are found to exist in the *Vedic* literature. Scientific truths regarding electricity, magnetism and formation of water, etc., as well as mathematical principles were, too, well known. The problem of construction of different types of altars and fire places gave rise to algebraic and geometrical principles. Enumeration up to such huge numbers as billions was in practice and words for numerals were used. Principles of calculus were also applied in those days of antiquity. Especially the *Atharva Veda* (*Pariśiṣṭa*) contains a number of *sūtras* (at least sixteen) and *upsūtras* (at least thirteen) which apply to and cover each and every branch of mathematics including arithmetic, geometry, trigonometry, conics, astronomy, calculus-diferential and integral. The late Śaṅkarācārya, Śrī Bhāratī Kṛṣṇa Tīrthaji Mahārāja (1884-1960) of Govardhan Matha, Puri, who was a gifted scholar, profound mathematician and a scientific thinker, explained and expounded the contents of the *sūtras* and contended that calculus comes in at a very early stage in these *Vedic* mathematics *Sūtras*[2]. Some of the important applications relating to differential as well as integral calculus as explained by him are given below :-

One of these *Sūtras* is *Gaṇaka Samuccaya* (i.e., the sum of the factors) which postulates that in every quadratic expression (in its standard form with 1 as the coefficient of x^2) the sum of its binomial factors is its first differential[3]. Let us take a quadratic expn. $X^2-5x-6= (x-2)(x-3)$. We, at once, with the help of this *sūtra*, say that its first differential, i.e., $D_1=x-2+x-3=2x-5$.

The first differential (of each term) can also be obtained by multiplying its Dhvaja ghāt (i.e., the power) by the Aṅka (i.e., its coefficient) and reducing it by one[4]. In the quadratic expn.x^2-5x-6, x^2 gives $2x$, $-5x$ gives -5 and 6 gives zero. $\therefore D_1=2x-5$. Similarly in $x^2-11x+10$, $D_1=x-10+x-1=2x-11$.

Again the same *Guṇaka Samuccaya sūtra* implies the same mathematical truth in the case of differential of a product of two variables, i.e., if $y=u.v$,, where u and v are functions of x, $\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$. The following examples will surffice to show the internal relationship subsisting between the factors of a polynomial and the successive differentials of that polynomial and also to show how easily on knowing the former, we can derive the latter and vice-versa :[5]

(i) In x^2+3x+2, $D_1=2x+3=x+1+x+2=\sum a$, where $x+1=a$ and $x+2=b$.

(ii) In $x^3+6x^2+11x+6$, $D_1=3x^2+12x+11=x^2+3x+2+x^2+5x+6+x^2+4x+3$
$=ab+bc+ca=\sum ab$.
and $D_2=6x+12=2(3x+6)=2(x+1+x+2+x+3)=2(a+b+c)=2\sum a=\underline{|2}\sum a$.

There is another relationship where in factorisation and differentiation are closely connected with each other and that is with regard to use of successive differentials for the detection of repeated factors. The following examples will serve to show the modus operandi in question[6] :-

Factorise (i) x^3-4x^2+5x-2

$D_1=3x^2-8x+5=(x-1)(3x-5)$

judging from the first and the last coefficients of the given expn. we can rule out 3x-5 and keep our eyes on $(x-1)$.

$D_2=6x-8=2(3x-4)$ $\therefore$ we have $(x-1)^2$.

$\therefore$ According to the Adyam Ādyena Sūtra, the given expn.$=(x-1)^2(x-2)$.

(ii) In $4x^3-12x^2-15x-4$, $D_1=12x^2-24x-15=3(4x-8x-5)=3(2x-5)(2x+1)$.

$D_2=24x-24=24(x-1)$. $\therefore$ As before we have $(2x+1)^2$

$\therefore$ The given expn. $=(2x+1)^2(x-4)$.

(iii) In $x^4-5x^3-9x^2+81x-108$, $D_1=4x^3-15x^2-18x+81$,

$D_2=12x^2-30x-18=6(2x^2-5x-3)=6(x-3)(2x+1)$,

$D_3=24x-30=6(4x-5)$

$\therefore$ The given expn. $=(x-3)^3(x+4)$.

One of the most important propositions in connection with the solution of a quadratic eqn. by calculus method is 'Calita Kalita Vargo Vivecakaḥ'. The *sūtra* state that the square of the first different in 1 coefficient of the quadratic expn. is equal to its discriminant[7]. Let us take a quadratic eqn. $7x^2-5x-2=0$. Its first differential coefficient is $14x-5$. From the *sūtra* $(14x-5)^2=25+56=81$, i.e., $14x-5=\pm 9$ $\therefore x=1$ and $-{}^2/_7$. Here the quadratic eqn. is broken down at sight into two simple equations which immediately give us two values of x, the unknown quantity. This calculus method for the solution of a quadratic eqn. is perfectly general and applies to all cases. The current method for the solution of $ax^2+bx+c=0$, by completing the square is clumsy. Another Indian method by *Sridharacarya* is a bit better than this, but the above mentioned *Vedic* method is very easy and rapidly quick. In the case of the quadratic eqn. $ax^2+bx+c=0$,

$D_1=2ax+b=\pm\sqrt{b^2-4ac}$, i.e., $2ax=-b\pm\sqrt{b^2-4ac}$ $\therefore x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Many other such applications are available from, the *Vedic Sūtras* relating to differential calculus. The late *Śaṅkarācārya* even goes on to remark that there are *Vedic* Sūtras which deal with and cover Leibnitz's theorem, Maclaurin's theorem and Taylor's theorem as well[8]. But these are not found to exist in his only available work, '*Vedic* Mathematics', though he claimed to have written sixteen volumes on these sixteen sūtras.

There are again some *Vedic sūtras* which make us believe that the process of integration was also known to the people of ancient India. One such sūtra is Ek ādhikena *Pūrveṇa* (i.e. by one more than the previous one) which states that in order to find the integral of a power of x, we add unity to the *pūrva* (i.e., the original index) and divide the coefficient

by the new index (i.e., the original one plus unity)[9], e.g., we have to integrate $\int 5x^4 = 5/5\, x^5+c = x^5+c$. The *Vedic* method is the same as the modern one is, but it does not mention the constant of integration. Moreover, in the case of integration of a fractional function the *Vedic Sūtra* is very useful. We know that the current procedure for breaking a fraction into partial fractions is very cumbrous, but the *Sūtra Parāvartya Yojayeta* (i.e., transpose and adjust) tackles the problem very quickly.[10]

we have to integrate $\dfrac{7x-1}{6x^2-5x-1}$

Here, $\dfrac{7x-1}{6x^2-5x+1} = \dfrac{7x-1}{(2x-1)(3x-1)} = \dfrac{A}{2x-1} + \dfrac{B}{3x-1}$.

The Parāvartya Yojayeta states that for getting the value of A equate its denominator to zero and substitute this value of X in the apn. without the factor which is A's denominator on the RHS, i.e., $2x-1=0 \therefore x=1/2$

$$\therefore A = \frac{7x-1}{3x-1} = \frac{7/2-1}{3/2-1} = 5.$$

$$\text{Similarly } B = \frac{7x-1}{2x-1} = \frac{7/3-1}{2/3-1} = -4. \therefore \frac{7x-1}{6x^2-5x+1} = \frac{5}{2x-1} - \frac{4}{3x-1}$$

$$\therefore \int \frac{7x-1\, dx}{6x^2-5x+1} = 5\int \frac{dx}{2x-1} - 4\int \frac{dx}{3x-1} = \frac{5}{2}\int \frac{d(2x)}{2x-1} - \frac{4}{3}\int \frac{d(3x)}{3x-1}$$

$$= 5/2 \log (2x-1) - 4/3 \log (3x-1) + c.$$

In the same way, $\dfrac{3x^2+12x+11}{(x+1)(x+2)(x+3)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{C}{x+3}$.

$$\therefore A = \frac{3x^2+12x+11}{(x+2)(x+3)} = \frac{3-12+11}{1\text{x}2} = 1,\ B = \frac{3x^2+12x+11}{(x+1)(x+3)} = \frac{12-24+11}{-1\text{x}1} = 1$$

$$\text{and } C = \frac{3x^2+12x+11}{(x+1)(x+2)} = \frac{27-36+11}{-2\text{x}-1} = 1 \text{ and so on.}$$

Thus with this *Vedic* method a laborious work of deriving and solving three simultaneous equations is totally avoided.

Thus the applications of *Vedic Sūtras* are numerous and and splendidly useful. These *Sūtras* and their expositions make us believe that process of intergration and principles of differentiation were known to the ancient Hindus. Some scholars, though, doubt the existence of these *Sūtras* in the Atharva Veda *Parisiṣṭa*, but the scholarship and profoundity of the late Śankarācārya cannot be doubted.

In the later period the idea of calculus originated in course of determining (i) the instantaneous motion of a planet, (ii) the position angle of the ecliptic with any secondary to the equator, (iii) the surface and volume of sphere and (iv) the value of π. The calculations of the instantaneous motion and position angle became necessary in connection with some of the astronomical problems, viz, computation of solar and lunar eclipses, ayana-Valana, etc. Ancient Indian astronomers from *Āryabhaṭa* I (476 A.D.) onwards attempted to solve such problems. Muñjāla (932 A.D.) was the first ancient astronomer to state the formula $d(\sin\theta)=\cos\theta.\, d\theta$, [11] but we cannot definitely speak of the method employed by him for finding this result. The formula occurs also in the works of *Āryabhata* II (950 A.D)[12], *Bhāskara* II (1114 A.D)[13] and later indian writers. There are sufficient grounds to show that *Bhāskara* II was the first mathematician to have indicated the method of obtaining the differentiation of $\sin\theta$ which is almost the same as attempted by the modern scholars. He defines the Sthula gati (rough motion) as well as Sūkṣma gati (instantaneous motion) of a planet, the latter being no other than the differentiation of the infinitesimal of a planet. He conceived of units which were small quantities of higher orders and used infinitesimal length and time which tended to zero incalculating the instantaneous motion of a planet. Moreover, he calculated the change in the sine-chord (Bhuja-Jyā) and cosinechord (Koṭi-Jyā) for small differences in the are from which $d\,(R\,\sin\theta) = \frac{R\cos\theta}{R}\,d\theta$ which corresponds to the modern formula- $d(\sin\theta)=\cos\theta.d\theta$. Again this formula occurs in his method of constructing a sine-table.[14] He has used it in solving the problems of

correction for the equation of centre, and also for finding complement of an angle. It has further been shown by *Bhāskara* that when a variable attains the maximum value, its differential vanishes[15], and when a planet is either in apogee or in perigee, the equation of centre vanishes[16] and hence for some intermediate position the increment of the equation of the centre (i.e.the differential) also vanishes. These statements force us to conclude that the mean value theorem of the differential calculus was known to him.

It is also gathered that some of the later Indian scholarsviz., Nīlkaṇṭha (1500 A.D.), Acyuta (1552-1620 A.D.) and others have also used the differentiation of such functions in their astronomical texts. Besides these, Nīlkaṇṭha made use of a result which involves the differentiation of an inverse sine function[17] and in the writings of Acyuta we find the differention of a quotient also[18].

As far as the development of integral calculus in India is concerned, the Indian method especially the method of *Bhāskara* for computing the surface and volume of a sphere included the process of integration. To find the area of the surface of a sphere *Bhāskara* invented two methods[19] which are almost the same as we employ these days for the same purpose. Again *Bhāskara's* rationale for the expression of the volume of a sphere is also the same as used in modern times.[20],Symbolically area of a circle $= \pi r^2$, surface of a sphere $= 4\pi r^2$ and volume of a sphere$=4/3\pi r^3$. These reusIts are in good agreement with modern values and the methods followed are allied to modern methods of integration.[21] *Bhāskara* was conscious of the fact that units must be as small as possible for their summation to approach the correct value, but he was not aware of the method of summation of an infinite series, i.e., the limit of a sum. However, the credit must be given to him for having used the method of integration although in a crude form. *Bhāskara* was followed by some of the later Indian mathematicians. Values for the surface and volume of a sphere as given by *Bhāskara* are found to exist in the works of *Nārāyaṇa Paṇḍita*.

Moreover, the idea of the summeation of an infinite series is first found in the *Yuktibhāṣā* of the 16th century.[22]

From the above considerations it may now be contended that principles of calculus-differential and integral, were used and applied in India as early as in the *Vedic* period. Later on, the method for the differentiation of some trigonometrical functions and also the process of some crude integration were conceived and thus the scholars of ancient India did some spade work in the field. However, a systematic and scientific study for the proper assessment of Hindu achievements in the field of calculus.

References

1. Bāpūdeo Śāstrī, *Bhāskara's* knowledge of differential calculus, J.A.B.S., Vol.27(no.3), 1858, p.213-16; P.C. Sengupta, Infinitesimal Calculus in Indian methematics-its origin and development, J.Deptt of letters, Calcutta University, vol. XXII, 1932, p1-17; and B.B. Datta and A.N. Singh, use of Calculus in Hindu mathematics, I.J.H.S., vol.19(2), 1984, Calcutta, p.95-104.
2. Ct. Jagadaguru *Śaṅkarācārya* , *Vedic* Mathematics, Varanasi, 1965.
3. Ibid, p.157.
4. Ibid, p.157.
5. Ibid, p.182.
6. Ibid, p.183.
7. Ibid, p.158.
8. Ibid, p.182.
9. Ibid, p.191.
10. Ibid, p.192
11. *Laghu-Mānasa*, II.7.
12. *Maha-Siddhanta*, III.15.
13. *Si, Śi Graha. Spaṣṭadhikara*, 36-37 and for the ralionale of the formula, B. Chaudhary, Artical study of mathematical contributions of Bhāskarācārya, Ph.D. thesis, 1989, L.N. Mithila University, Darbhanga, p.447-50.
14. *Si.Śi. Golādhyāya*, Jyotpati.21.
15. *Si.Śi. Graha. Spaṣṭādhikāra*,31 (Vasana).
16. Ibid, 39 (Vāsaṇā)
17. Commentary on Aryabhatiya, II.12 and *Tantra-Saṁgraha*, II. 53-54.
18. Sphuta-Nirnaya-Tantra, III.19-20 and Karnottama, II.7.
19. *Si.Śi.* Gotādhyāya, Bhuvana.55-57 (Vāsanā).
20. *Ibid*, 58-61 (Vāsanā),
21. *Cf.B. Chaudhary*, p.cit., p.461-62.
22. *cf. c.n. Srinivasiengar*, The history of ancient Indian mathematics, Calcutta, 1967, p.148ff.

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PARETO OPTIMUM

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Abstract

In this note we present to the nonexperts the Pareto philosophy which is popularly known as "everybody gains and nobody loses and show some applications to economy". The material of this paper have been selected from [9], [10] and [11] and can be considered as a brief account on Pareto optimum.

INTRODUCTION

Nowadays the terms, Pareto optimum, Pareto efficient, equilibrum are often found in articles of economics, mathematical economics, games theory, psychology, investment theory, theory of finances, traffic problems, problems of human migration and other related areas. First let see what is meant by Pareto optimum.

In articulating his long chapter on economic equilibrium, Pareto [4, p.108], remarked that economic equilibrium can be defined in different ways which come to the same thing in the end. Subsequent researches in the field have shown how accurate he was a century ago. Pareto's special contribution is his description of optimal ophelimity (commonly known as Pareto optimum). Pareto in his book, Manuel d' 'economie politique ([4], [1909] and [1971, p.261]), describes the economic equilibrium and remarks that

.., the members of a collectivity enjoy maximum ophelimity in a certain position when it is impossible to find a way of moving from that

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position very slightly in such a manner that the ophelimity enjoyed by each of the individuals of that collectivity increases or decreases. That is to say, any small displacement in departing from that position necessarily has the effect of increasing the ophelimity which certain individuals enjoy, and decreasing that which others enjoy, of being agreeable to some and disagreeable to others.

An allocation is specified by the consumption levels of each consumer and the input and output levels of each producer. An allocation is said to be Pareto optimal (efficient) if it is not possible to organize the production and distribution so as to increase the utility of one or more individuals without causing loss to the utility of others. Edgeworth [3] and Pareto [4, p.534] considered the relation between competitive equilibrium and optimal allocations by starting from the latter. The concept of Pareto optimum is used nowadays in almost all branches of economics, investments theory, traffic problems, problems of human migration and so on.

ECONOMY AND RELATED CONCEPTS

Now in order to give mathematical formulation of the above Pareto's philosopy we need an economic model, known as Arrow-Debreu model of economy. An economy is described by :

m consumers indexed by $i = 1,2, \ldots, m$;
n producers indexed by $j = 1,2, \ldots, m$;
a consumption set $(X_i, \underset{i}{\preceq})$ where X_i is a nonempty subset of R^l and $\underset{i}{\preceq}$ is a preference preordering, i.e. a reflexive and transitive relation on X_i; for each $j = 1,2 \ldots, n$, a nonempty subset Y_j, of R^l, called the production set for the producer j, and an a priori given vector $\omega \in R^l$, called the total resources of the economy ε.

A pair $(x, y) = ((x_i), (y_j))$ where $x_i \in X_i$ and $y_j \in Y_j$, $i = 1,2,\ldots, m$ and $j = 1,2,\ldots, n$ is called a state of the economy ε. Thus a state of the economy ε is an $(m+n)$ tuple R^l which can be represented by a point of

$R^{(m+n)l}$. Given a state $(x, y)=((x_i), (y_j))$ of ε, the point $x - y = \sum_{i=1}^{m} x_i - \sum_{j=1}^{n} y_i$ is called the net demand and the point $z = x - y - \omega$ is called the excess demand. Thus every point of the set $Z = X - Y - \{\omega\}$ represents an excess demand corresponding to a state, where $X = \sum_{i=1}^{m} x_i$ and $Y = \sum_{j=1}^{n} y_j$

A state $(x,y) = ((x_i), (y_j))$ of ε is called a market eqilibrium if $x - y = \sum_{i=1}^{m} x_i - \sum_{j=1}^{n} y_i = \omega$, that is if excess demand is 0. A state $(x, y) = ((x_i), (y_j))$ of ω is said to be attainable if $x_i \in X_i$ for $i = 1,2,\ldots, m$, $y_j \in Y_j$ for $j=1,2,\ldots, n$ and $x - y = \sum_{i=1}^{m} x_i - \sum_{j=1}^{n} y_j = \omega$. Note that the two different notations $x = (x_i)$ and $y = (y_j)$ and $x = \sum_{i=1}^{m} x_i$ and $y = \sum_{j=1}^{n} y_j$ have been used.

The set of all attainable states of ε is denoted by A. If X_i is the commodity space, a subset of R^l and $\underset{i}{\preceq}$ is the preference preordering of the i th consumer, an increasing function $u_i : x_i \to R$ (i.e. $x_i, x'_i \in X_i$ with $x_i \underset{i}{\preceq} x'_i \to u_i(x_i) \leq x_i(x'_i)$) is called the utility function (ophelimity in Pareto's term) of the ith consumer. Without loss of generality we may assume u_i to be nonnegative.

Let us consider the following condition on $(X_i, \underset{i}{\preceq})$:

(a) for each $x'_i \in X_i$, the set $\{x_i \in X_i : x_i \underset{i}{\preceq} x'_i\}$ is closed.

It is well known that $\underset{i}{\preceq}$ satisfying (a) represents an increasing upper semicontinuous utility function $u_i : X_i \to R$. Conversely given a upper semicontinuous function $u_i : X_i \to R$, we can define a preordering on $\underset{i}{\preceq} x_i$ as follows : with x_i and x'_i in X_i, $x_i \underset{u_i}{\preceq} x'_i$ if and only if $u_i(x_i) \leq u_i(x'_i)$. Clearly $\underset{u_i}{\preceq}$ satisfies (a) above and u_i is increasing with respect to $\underset{u_i}{\preceq}$. This provides a passage from an economy $\mathcal{E} = ((X, u_i), (Y_j))$ to an economy $\varepsilon = \left((X_i, \underset{i}{\preceq}), (Y_j)\right)$ and viceversa.

Throughout the rest of the paper we will assume that the economy ε is given by $\varepsilon = \left((X_i, \underset{i}{\preceq}), (Y_j)\right)$ where $\underset{i}{\preceq} = \underset{u_i}{\preceq}$ and u_i is upper

semicontinuous and concave for each $i = 1,2,\dots, m$. That is, each preference can be represented by an upper semicontinuous concave function.

Private ownership economies (The Arrow Debreu Model).

A private ownership economy is defined by $\varepsilon = \left((X_i, \underset{i}{\preceq}), (Y_j), \omega\right)$ where for each i, there is a point ω_i (resources of the i th consumer) of R^l such that $\omega = \sum_{i=1}^{m} \omega_i$

Pareto optimum

Let us consider an economy $\varepsilon = \left((X_i, \underset{i}{\preceq}), (Y_j), \omega\right)$. We define a preordering $\underset{a}{\preceq}$ on A, the set of all attainable states of ε as follows : Given two attinable states or allocations $((x_i), (y_j))$ and $((x'_i), (y'_j))$ $((x_i), (y_j)) \underset{a}{\preceq} ((x'_i), (y'_j))$ if and only if $x_i \underset{i}{\preceq} x'_i$ for each i, i.e. each consumer i desires his consumption x'_i at least as much as his consumption x_i. Given two comparable attinable states $((x_i), (y_j))$ and $((x'_i), (y'_j))$, the second one is said to be better or more efficient than the first one, if $x_i \underset{i}{\preceq} x'_i$ for each i and $x_i \underset{i}{\prec} x'_i$ for at least one i. In this case we write $((x_i), (y_j)) \underset{a}{\prec} ((x'_i), (y'_j))$. It should be noted that two attainable states $((x_i), (y_j))$ and $((x'_i), (y'_j))$ may be incomparable with respect to $\underset{a}{\preceq}$.

A maximal element of A with respect to $\underset{l}{\prec}$ is called a Pareto optimum of the economy ε.

(A point $\bar{x}_i$ is a station point if there is no consumption $x'_i \in X_i$ such that $x'_i \underset{i}{\prec} x_i$, i.e. $\bar{x}_i$ is a greatest element of X_i with respect to $\underset{i}{\preceq}$ which is a complete preordering.)

An equilibrium point of the private ownership economy ε is $(m+n+1)$-tuble $((\bar{x}_i), (\bar{y}_j), \bar{p})$ of points of R^l such that :

(i) for each $i = 1,2, \dots, m$, $\bar{x}_i$ is a greatest element of $\{x_i \in X_i : \bar{p} \cdot x_i \leq \bar{p} \cdot w_i + \sum_{j=1}^{n} \; i, j\, (p.y_j)\}$ with respect to $\preceq i$;

(ii) for each $j = 1,2, \ldots, n$, $\bar{y}_j$ maximizes the profit relative to $\bar{p}$ on Y_j; and

(iii) $\bar{x} - \bar{y} = w$ where $\bar{x} = (\bar{x}_i)$ and $\bar{y} = (\bar{y}_j)$ and $w = \sum_{i=1}^{m} w_i$ (For economic interpretation see Debreu [2.p.79]).

Debreu [2] in his book has proved the existence of an equilibrium $((\bar{x}_i), (\bar{y}_j), \bar{p})$ of a private ownership economy (commonly known as Arrow-Debreu model of economy) and has shown that given an equilibrium $((\bar{x}_i), (\bar{y}_j), \bar{p})$ of the economy $\mathcal{E}$, $((\bar{x}_i), (\bar{y}_j))$, is a Pareto optimum if x_i is not a satitation point and given a Pareto Optimum $((\bar{x}_i), (\bar{y}_j))$, of ε, there is a price system p such that $((\bar{x}_i), (\bar{y}_j), \bar{p})$ is an equilibrium point of $\mathcal{E}$ provided x_i is not a satiation point, where $((\bar{x}_i), (\bar{y}_j))$ is an allocation and p is a price system. These two theorems are known as the two fundamental theorems of mathematical economics. Arrow's and Debreu's work on economic equilibria are and will be considered as landmarks in the theory of economic equilibria.

In the paper [11] we have proved the same two theorems by different approach. Debreu first proved the existence of equilibrium and then showed that an equilibrium is equivalent to a Pareto optimum certain condition, while we proved the existence of a Pareto optimum and equilibrium. Debreu used Kakutani's fixed point theorem, while we have not used any fixed point theorem. However, we should point out that to prove the equilibrium first one has to use some kind of fixed theorem.

In order to explain our method employed in [11] we choose for simplicity the pure exchange economy described below :

PURE EXCHANGE ECONOMY

The pure exchange economy is the economy where the production sector is inactive, i.e., $Y = \sum_{j=1}^{n} y_j = \{0\}$.

The model of pure exchange economy ε is described by m consumers indexed by $i = 1,2, \ldots, m$; the commudity space $\Omega = \{(x^1, x^2, \ldots, x^l) \in R^l\}$ which is a positive cone of R^l, the space of all state of economuy $A = \Omega^m = \{x = (x_1, x_2, \ldots, x_m) \in R^{lm} : x_i \in \Omega, i = 1,2, \ldots, m$ and $\sum_{i=1}^{m} x_i = w\}$ where $w \in R^l$ is the total resources of the economy and for each $i = 1,2,\ldots,m$, $u_i : \Omega \to R_+$ is a continuous function called the utility for the i -th consumer. Each element $(x_1, x_2, \ldots, x_m)$ is called an attainable state. As above, given two attainable states (x_i) and (x'_i), $(x_i) \preceq a\ (x'_i)$ if and only $x_i \preceq_i x'_i$, i.e., $u_i(x_i) \leq u_i(x'_i)$ for each $i = 1,2, \ldots, m$. Pareto optimum of the economy is a maximal element of A with respect to $\preceq a$. Now let us define the function $f : A = \Omega^m \to R_+$ by

$$f(x) = f(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{m} u_i(x_i)$$

for each $x = (x_1 \ldots, x_m) \in A$. Thus $f(x)$ measures the total utility or satisfaction for the consumers on the state $x \in A$. Clearly f is continuous function defined on the compact set A and hence $M = \sup_{x \in A} f(x)$ is achieved. Now it is easy to see that each point $\hat{x} \in f^{-1}(M)$ is a Pareto optimum of the pure exchange economy ε. Indeed if $\hat{x} \in A$ is not a Pareto optimum, i.e., not maximal element of A with respect to $\preceq a$, then there must be an element $\bar{x} = (\bar{x}_i) \in A$ such that $\hat{x} \prec_a \bar{x}$, i.e., $u_i(\hat{x}_i) \leq u_i(\bar{x}_i)$ for all i and $u_i(x_i) < u_i(x_i)$ for at last one i. This means that $M = f(\hat{x}) = \sum_{i=1}^{m} u_i(\hat{x}_i) < \sum_{i=1}^{m} u_i(\bar{x}_i) = f(\bar{x})$ which is a contradiction. Hence $\hat{x}$ is a Pareto optimum.

It is also interesting to note that the total utility or the total satisfaction of all the consumers is the same at each Pareto optimum.

PARETO SOLUTION OF A CONE INEQUALITY

A nonempty subset P of a real Banach space V is called a **cone** if $\bar{P} = P$, $P+P \subset P$, $R_+P \subset P$ and $P \cap (-P) = \{0\}$, where P is is the closure of P and $R_+ = [0,\infty)$.

Each cone P induces in V an ordering $\preceq$ defined by $x \preceq y$ if and $y - x \in P$. This relation $\preceq$ is evidently reflexive, antisymmetric and transitive. The pair (V,P) is called an ordered Banach space with the ordering $\preceq$ induced by P, the positive cone of V. $P^* = \{f \in V^* : f(x) \geq 0$ all $x \in P\}$ is called the dual cone where V^* is the continuous dual of V, that is P^* is the set of order preserving continuous linear functional on V.

A cone P of V is said to be **normal** if and only if there exists a positive number e such that if $x,y \in P$ with $||x|| \geq 1$ and $||y|| \geq 1$, then $||x+y|| \geq e$ (for other equivalent definitions see [4] and [5]). In what follows V will always denote an ordered Banach space and we write $x \prec y$ if $y - x \in P \setminus \{0\}$.

Let X be a nonempty convex subset of a Hausdorff real topological vector space. A mapping $F:X \to V$ is said to be **order or cone convex** if $F(\lambda x+\mu y) \preceq \lambda F(x)+\mu F(y)$ for all $x,y \in X$ and $\lambda \geq 0$, $\mu \geq 0$ with $\lambda+\mu = 1$, i.e. for such x,y, λ, μ we have

$$\lambda F(x) + \mu F(y) - F(\lambda x+\mu y) \in P.$$

We note that F is cone convex if and only if $f.F$ is convex for all $f \in P^*$. [If F is cone convex, then $f.F$ is convex for $f \in P^*$ as f is order preserving. Next, let $f\,F$ be convex for each $f \in V^*$. If possible, let for some $x,y \in X$ and $\lambda \geq 0$, $\mu \geq 0$ with $\lambda + \mu = 1$, $[\lambda F(x)+\mu F(y)-F(\lambda x+\mu y)] \notin P$, then by a consequence of the separation theorem (see p.225 [3]) there exists a $f \in P^*$ such that $f\,[\lambda F(x)+\mu F(y)-F(\lambda x+\mu y)] < 0$ which implies that $f\,F$ is not convex].

A mapping $F : X \to V$ is said to be **order or cone lower (upper) semicontinuous** if $f.\,F$ is lower (upper) semicontinuous for each $f \in P^*$.

We recall from [6] and [8] that a function $G : X \times X \to \mathbb{R}$ is called ***monotone*** if $G(x,y) + G(y,x) \geq 0$ for all $x,\; y \in X$ and is called **hemicontinuous** if the function $k(t) = G(x+t\,(y-x),y))$ of the real variable

$t \in [0,1]$ is lower semicontinuous on X as $t \downarrow 0^+$ for arbitrary given vectors x and y in X.

A mapping $F : X \times X \to V$ is said to be **order or cone monotone** if $F(x,y) + F(y, x) \succeq 0$ for all $x,y \in X$ and is said to be **strictly cone monotone** if an addition $F(x,y) + F(y, x) = 0$ implies $x = y$. A mapping $F: X \times X \to V$ is said to be **order or cone hemicontinuous** if $f\ F$ is hemicontinuous for each $f \in P^*$.

With X a nonempty convex subset of a real Hausdorff topological vector space E, (V, P) an ordered Banach space and P a normal cone, we consider the following order or cone variational problem (to see connection with variational problem we refer to our work [8, 1990)]).

We consider the following problem :

(3.1). $F: X \times X \to V$ is a mapping with $F(x, x) = 0$ for each $x \in X$ and satisfying

(i) for each $x \in X$, $F(x,.)$ is (a) cone concave and (b) cone upper semicontinuous on E;

(ii) F is cone monotone and cone upper hemicontinuous; and

(iii) for $x, y \in X$, $F(x, y) \in [P \cup (-P)]^c$ if and only if $F(y, x) \in [P \cup (-P)]^c$

where A^c denotes the complement of the set A.

We are interested to find the existence of a point $x_0 \in X$ such that

$$F(x_0, y) \in (-P) \cup [P \cup (-P)]^c \text{ for all } y \in X. \qquad (3.1)$$

x_0 will be called *a* Pareto solution of the variational cone inequality (3.1).

The following theorem is proved in ([**9**], Corollary 2.2')].

Theorem 3.1. *If X is a nonempty compact convex subset of a Hausdorff real topological vector space E and $F : X \times X \to V$ is a mapping such that for all $x, y \in X$, $F(x, y) = -F(y, x)$* and for each $x \in X$, $F(x, .)$ *is cone* ... *and satisfies the condition :*

(a) $\bigcup_{y \in X} \{y \in X : F(x,y) \prec 0\} = X$ implies $\bigcup_{x \in X} [B(x)^c] = X$,

where $B(x) = \overline{\{y \in X : F(x, y) \prec 0\}}^c$, then there exists a Pareto solution of the cone variational inequality (3.1).

GENERALIZED PARETO OPTIMUM

In this section we introduce the concept of Pareto optimality of a mapping. if a point $x_0 \in X$ satisfies the cone inequality

$$F(x_0, y) \in P \cup [P \cup (-P)]^c \text{ for all } y \in X. \quad (3.1)$$

then x_0 is said to be a Pareto maximal of F and if $x_0 \in X$ satisfies the cone inequality

$$F(x_0, y) \in (-P) \cup [P \cup (-P)]^c \text{ for all } y \in X. \quad (3.2)$$

then x_0 is said to be a Pareto minimal of F. In either case x_0 is said to be a generalized Pareto optimum, or simply a Pareto optimum of F. Throughout the rest of the paper by a generalized Pareto optimum or Pareto optimum we always mean a Pareto maximal. In order to apply our concept to more specific problems such as the problems of economics and game theory we need to formalize further our concepts.

With a mapping $f : X \to V$, we can define a preordering $\underset{f}{\preceq}$ in X as follows :

for $x, y \in X$, $x \underset{f}{\preceq} y$ if and only if $f(x) \preceq f(y)$ where $\preceq$ is the order in V induced by the cone P. Thus $\underset{f}{\preceq}$ is reflexive and transitive as $\preceq$ is so. Further we define $x \underset{f}{\prec} y$ if and only if $f(x) \prec f(y)$. Obviously the mapping $f : (X, \underset{f}{\preceq}) \to (V, \preceq)$ is an increasing mapping.

A point $\bar{x} \in X$ is called a maximal element with respect to $\underset{f}{\preceq}$ if there is no point $x \in X$ such that $x \underset{f}{\prec} y$. It readily follows that $\bar{x} \in X$ is maximal element with respect to $\underset{f}{\preceq}$ if and only if $[f(\bar{x}) - f(y)] \in [P \cup (-P)]^c$ for all $y \in X$, that is if and only if x is a Pareto solution of the cone inequality $F(x, y) \in P \cup [P \cup (-P)]^c$

A maximal element $\bar{x} \in X$ with respect to $\underset{f}{\preceq}$ is said to be generalized Pareto optimum or simply a Pareto optimum for f.

We note that if x_1 and x_2 are two Pareto optima of f, then x_1 and x_2 are either incomparable or indifferent.

The next theorem is the Corollary 3.2 in [**9**].

Theorem 4.2. *Let X be compact convex subset of E and $f : X \to V$ be a cone upper semicontinuous and cone concave mapping satisfying the condition (a) of Theorem 3.1 for F defined by $F(x, y) = f(y) - f(x)$, $x, y \in X$. Then there is a Pareto optimum x_0 of f.*

Proof. We set $F(x, y) = f(y) - f(x)$, $x, y \in X$ and apply Theorem 3.1 to obtain the theorem.

APPLICATIONS TO ECONOMY

In [10] we have obtained the following two propositions :

Proposition 5.1. A point $x_0 \in X = A = \Omega^m$ of the pure exchange economy $\mathcal{E}$ if and only if x_0 is *a Pareto optimum of the mapping $f : X \to V$ defined as*

$$f(x) = (u_1(x_1),\ u_2(x_2),\ \ldots,\ u_m(x_m))$$

where $x = (x_1, x_2, \ldots, x_m) \in \Omega^m = X$, $x_i \in \Omega$ for each $i = 1, 2, \ldots, m$, $V = (\mathrm{R}^m, \Omega')$ *is an ordered Banach and* $\Omega' = \{(x_1, x_2, \ldots, x_m) \in \mathrm{R}^m : x_i \geq 0$ *for* $i = 1, 2, \ldots, m\}$ *is the positive cone of* R^m.

Proposition 5.2 Let $\mathcal{E} = ((X_i, \underset{i}{\preceq})\ (y_j), (w_i))$ *be a private ownership economy and $X = A$ be the set of all attainable states of $\mathcal{E}$. If $f : X \to \mathrm{R}^m$ be the mapping defined by*

$$f((x_i), (y_j)) = (u_1(x_1), (u_2(x_2), \ldots, u_m(x_m)),$$
$$((x_i), (y_j)) \in A = X.$$

Then $((\bar{x}_i), (\bar{y}_j)) \in A$ *is a Pareto optimum of f if and only if $((x_i), (y_j))$ is a Pareto optimum of $\mathcal{E}$.*

For further results on mathematical economics see [**9**] and [**10**].

ACKNOWLEDGEMENT

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REFERENCES

1. K. J. Arrow and P. Hahn, *General Competitive Analysis*, Holden-Day Inc., San Francisco, 1971.
2. G. Debreu, *Theory of Value*, Wiley, New York, 1959.
3. F. Y. Edgeworth, *Mathematical Psychics*, C. kegan Paul, London, 1981.
4. G. Jameson, *Ordered Linear Spaces*, Lecture Notes 141, Springer-Verlag, New York, 1970.
5. J. L. Kelley and I. Namioka. *Linear Topological Spaces*, Von Nonstrand, Princeton, N.J., 1963.
6. U. Mosco, *Implicit variation problems and quasi variational inequalities*, Nonlinear Operators and Calculus of Variations ((by J. P. Gossez, E. J. Lami Dozo, J. Mawhin, L. Waebroek, ed.), Lecture Notes in Math., vol. 543, Springer-Verlag, New York, 1976, pp.83-156.
7. V. Pareto, *Manuel d'economic politique*, Girard and Briere, 1971, English Translation by A. S. Schwier, Augustus M. Kelley Publishers, New York, 1909.
8. E. Tarafdar, *Nonlinear variational inequality with application to the boundary value problem for quasilinear operator in generalized divergence form,* Funkcialaj Ekvaeioj 33 (1990), 441-453.
9. E. Tarafdar, *Pareto solution of cone variational inequality and Pareto Optimality of a mapping,* Proceedings of World Congress of Nonlinear Analysis, Tampa, Florida, August 19-26 (1992). Walter de Gruyter, Berlin, new York (1996), 2511-2519.
10. E. Tarafdar, *Applications of Pareto optimality of a mapping to mathematical economics,* Proceeding of the First World Congress of Nonlinear Analysis, Tampa, Floria, August 19-26, (1992). Walter de Gruyter, Berlin, New york (1996), 2431-2439.
11. E. Tarafdar, *Pareto optimum and equilibrium points of private ownership economies - a simpler approach without fixed point theorem*, Arab. J. Math Sc. 1 (1995), 65-74.

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट
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खण्ड Vol. 1 (1998), No.1, 130-131

वेदे वैमानिकी विद्या

वेद प्रकाश शास्त्री

विश्वप्रसृतानां सर्वासां सत्य विद्यानां बीजं वेदे परिलक्ष्यते। भौतिकी विद्या सामाजिकी विद्या आध्यात्मिकी विद्या चेति त्रिविधा विद्या भवन्ति। आस्वेव विलसन्ति सर्वा विद्याः। निखिलमपि प्रकृतिविज्ञानं भौतिक्यन्तर्गतम्। सामाजिकविद्यान्तर्गता सर्वा नीतयो लोकव्यवहृतयश्च। अन्तस्तत्वावबोधाय समपेक्षिताः सर्वा विद्या आध्यात्मिक विद्यान्तर्गता एव। सांसारिक सुखोपलब्धयर्थं भौतिकं विज्ञानमिष्यते तत्रापि वाणिज्यलाभाय यात्रा सुखाय च यातायात साधनानि गवेषणीयानि। आदिकालादेव मनुष्यैः समुद्रयाने विशेषाभिलाषः प्रकटितः। वेदेषु विमान विद्या विद्यते। प्राचीन भारते वेदादेव विज्ञानप्रकाशमवाप्य विद्वद्भिर्विमान विद्यां प्रसाराय ग्रन्था विरचिताः। सम्प्रति समग्रतया न लभ्यन्ते ते ग्रन्थाः। केचन विमानविद्याग्रन्था अंशतो दृक्पथमवतीर्णाः। तेष्वन्यतमो महर्षि भारद्वाज प्रणीतो यन्त्रसर्वस्वनामको ग्रन्थः सम्प्रति विदुषां बुद्धिगम्यः। अस्य ग्रन्थस्य वैमानिकप्रकरणस्य कतिपया भागा बोधानन्दवृत्त्यन्विताः सम्प्राप्ताः। महर्षि भारद्वाजः कथयति यद् वेदार्णवं निर्मथ्य मनुष्याणाम् अभीष्ट फलदायकं यन्त्रसर्वस्वनामकं ग्रन्थरत्नं प्रददामि। भारद्वाजो वेदे विमान विद्यां ददर्श। यथा-

निमर्थ्य तद् वेदाम्बुधिं भरद्वाजो महामुनिः।
नवनीतं समुद्धृत्य यंत्रसर्वस्वरूपकम्।।
प्रायच्छत् सर्वलोकानाम् ईप्सितार्थफलप्रदम्।
तस्मिन चत्वारिंशतिकाधिकारे सम्प्रदर्शितम्।।
नानाविमानवैचित्र्यरचनाक्रमबोधकम्।
अष्टाध्यायैर्विभाजितं शताधिकरणैर्युतम्।।
सूत्रैःपञ्चशतैर्युक्तं व्योमयानप्रधानकम्।
वैमानिकाधिकरणम् उक्तं भगवता स्फुटम्।।

विमानेच्छां मनुष्याणां मनसि प्रबोधयितुं यजुर्वेदीयो मन्त्रः प्रस्तूयते - समुद्रं गच्छ स्वाहाऽन्तरिक्षं गच्छ स्वाहा ।।यजु0 6/21 हे मनुष्य त्वं सुखप्रदं यानामारूह्य समुद्रयात्रां व्योमयात्रां च कुरूष्व। अद्यत्वे देशे विदेशे वा तीव्रगतिमन्ति यानानि धावन्ति। एवं विधानां विमानानां विज्ञानवर्णनं वेदेष्वस्ति। वैज्ञानिका वेदज्ञानसाहाय्येन अन्तरिक्षयात्राकरणाय मनोवेगसमम् अद्भुतं यानं निर्मातुमर्हन्ति।मनोवेगतुल्य विमाननिर्माण-प्रसंगे प्रस्तूयते वेदमन्त्रः-

आचार्य उपकुलपतिश्च, हरिद्वारस्थ गुरुकुल कांगड़ी विश्वविद्यालयस्य

आ वां रथं पुरुमायं मनोजुवंजीराश्वं यज्ञियं जीयसे हुवे।
सहस्त्रकेतुं वनिनं शतद्वसुं श्रुष्टीवानं वरिबोधामभिप्रयः।। ऋक् 1-139-1

अस्मिन् मन्त्रे यानि रथाविशेषणानि सन्ति तानि सारमयानि। पुरुमायं विशेषणपदं विलक्षणं बुद्धिचातुर्यं व्यनक्ति, मायेति प्रज्ञा नाम, पुर्व्या मामया प्रज्ञया सम्पादितमिति पुरुमायम्। मनोजुवं = मनोवद्वेगवन्तम्।श्रूष्टीवानम् = श्रुष्टीः क्षिप्रगतीर्वनति भाजयति यस्तम्। वरिवोधाम् = सुखसेव्यम् सुखसेवनकारयितारम्। इमानि विशेषणपदानि सामान्ययानस्य रथस्य वा न सन्ति अपितु विमानगतान्येव।

विमानाविषयक एकोमन्त्र ऋग्वेवे दृश्यते - यथा - ऋक् 3,58,8,
"रथो हव्यमृतजा अद्रिजूतः परिद्यावां पृथिवी याति सद्यः"।।

अस्मिन् मन्त्रे शिल्पसम्पादितस्य रथस्य वर्णनं विद्यते। अनेन वर्णितेन रथेन मनुष्याः सत्वरमेव पृथिव्या अन्तरिक्षस्य च यात्रां कुर्वन्ति। एतेन अश्विनौ यात्रां कुरुतः, महर्षिभाष्यानुसारं अश्विनौ पदं राष्ट्रपतियात्रायाः सेनापति यात्रायाश्च बोधकमस्ति अस्य मन्त्रस्य भावार्थे महर्षिणा लिखितमस्ति यत् विमानादियानान्यारुह्याभिष्टानि सुखानि प्राप्य यत्रेच्छा तत्र सद्यो गन्तुं शक्नुवन्ति। अर्थात् ये वैज्ञानिका अग्नि विद्युदादिभिर्विमाननिर्माणं कुर्वन्ति ते समीप्सितं सुखं प्राप्य इच्छापूर्वकं यत्र तत्र शीघ्रमेव प्रयान्ति। ऋग्वेदस्यैकस्मिन् वेद मन्त्रे नामपूर्वकं विमान वर्णनमस्ति। यथा-

सोभापूषणौ रजसो विमानंसप्तचक्रं रथमविश्वपिन्वम्।
विषूवृतं मनसा युज्यमानं तं जिन्वथो वृषणा पञ्चरश्मिम्।।ऋक् 2/40/3

अद्यत्वेऽपि तानि यानानि न दृश्यन्ते यानि वेदेषु वर्णितानि सन्ति। वैज्ञानिका वेदज्ञानसाहाय्येन तान्यपि निर्मातुं प्रभविष्यन्ति।

सारांश

विज्ञान वेत्ताओं ने इसी प्राचीन भारत में विमान विद्या के विषय के अनेक ग्रन्थ लिखे हैं। महर्षि भारद्वाज विरचित मन्त्रसर्वस्व नामक ग्रन्थ आज आंशिक रूप में उपलब्ध है। उसमें विमान निर्माण एवं परिचालन सम्बन्धी वर्णन है। उन्होंने मन्त्रसर्वस्व में लिखा है कि वेद रूपी सागर का मन्थन करने के बाद ही यह विज्ञान ग्रन्थ नवनीत के रूप में समाज को प्रदान किया है। वेदों में विमान विषयक ऐसे मन्त्र है जिनके चिन्तन से नाना प्रकार के विमान बनाए जा सकते है। भौमिक रथ जलीय रथ तथा आकाशीय रथ निर्माण का संकेत वेदों में मिलता है।

फार्म-4

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट

खण्ड 1 अंक 1, 1998

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मैं एस. एन. सिंह, कुलसचिव गुरुकुल कांगड़ी विश्वविद्यालय हरिद्वार घोषित करता हूँ कि उपरिलिखित तथ्य मेरी जानकारी के अनुसार सही हैं।

हस्ताक्षर

एस. एन. सिंह

कुलसचिव

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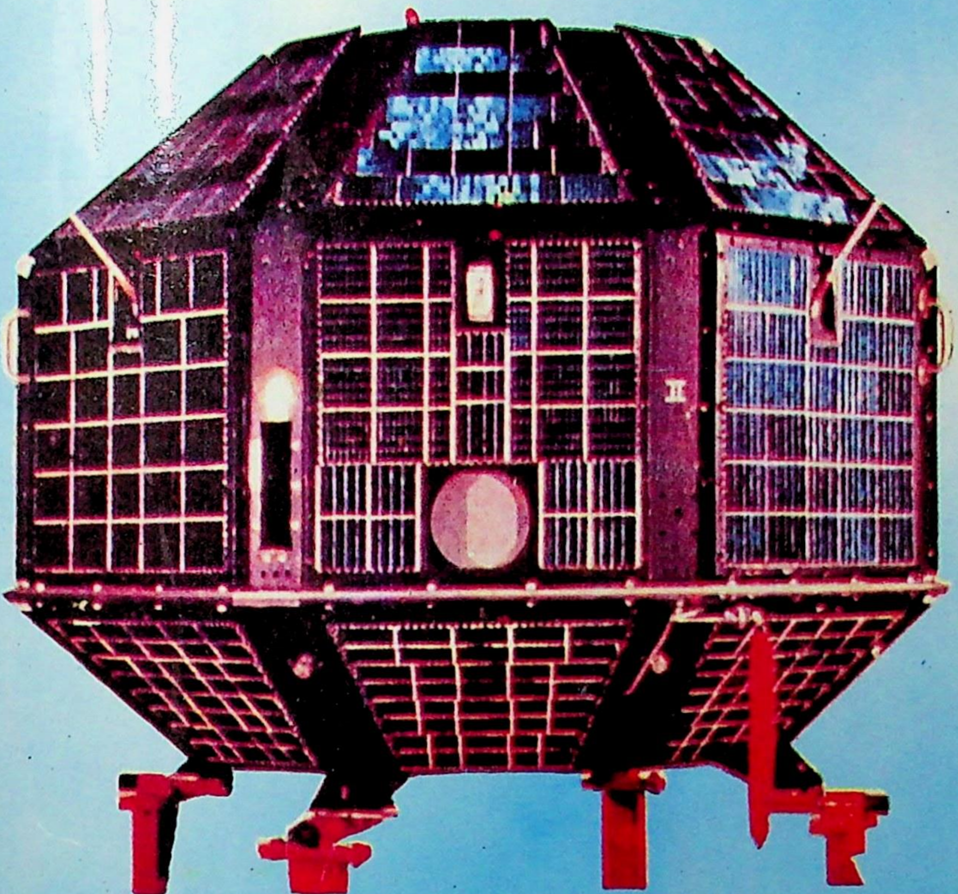
Gurukula Kangri Vijñāna Patrikā

आर्यभट

ĀRYABHAṬA



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Gurukula Kangri Vijñana Patrikā Āryabhaṭa

[illegible]

AN EXT[illegible]ON OF KY FAN'S BEST APPR[illegible]TION THEOREM

[illegible] SINGH**

1. Let [illegible] a nonempty subset of a normed [illegible] space X. Let [illegible] be a function. We look for [illegible] satisfying the following equation

$$\|x - f(x)\| = d(f(x), C) [illegible]$$

If a solution x in C exists, it is called a best approximation [illegible] We note that $f(x)$ is a solution of (*) [illegible] where [illegible] is the metric projection [illegible] list of reference Cheney [1], Lassonde [illegible] [10], Sehgal & Singh [11], Singer [12], [illegible] Wolters [15] are worth mentioning.

In 1969, Ky Fan [2] established the following theorem called the Ky Fan's best approximation theorem. This result is of great importance in nonlinear analysis [illegible] theorem in a more broad subset C of a Banach space [illegible] these results here for easy reference.

Theorem 1.1. Let C be a nonempty compact convex subset of a

[illegible]

15 1040 E

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट, खण्ड १ (१९९८), अंक २, १३३-१४६
Gurukula Kangri Vijñāna Patrikā Āryabhaṭa, Vol. 1 (1998), NO. 2, 133-146

AN EXTENSION OF KY FAN'S BEST APPROXIMATION THEOREM

JINLU JI* and S. P. SINGH**

1. Let C be a nonempty subset of a normed linear space X. Let $f: C \to X$ be a function. We look for an x in C that satisfies the following equation

$$\|x - f(x)\| = d(f(x), C) = inf\{\|y - f(x)\| : y \in C\} \qquad (*)$$

If a solution x in C exists, it is called a best approximation for $f(x)$. We note that $f(x)$ is a solution of (*) if and only if x is a fixed point of $P_C{}^0 f$ where P_C is the metric projection on C. For related work the list of reference Cheney [1], Lassonde [3], Park [5], Prolla [8], Reich [10], Sehgal & Singh [11], Singer [13], Singh & Watson [14] and Waters [15] are worth mentioning.

In 1969, Ky Fan [2] established the following Theorem called the Ky Fan's best approximation theorem and has been of great importance in nonliear analysis. Reich [9] generalized this theorem to more broad subset C of a Banach space. We state these theorems here for easy reference.

Theorem 1.1. Let C be a nonempty, compact, convex subset of a

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normed linear space X and let $f:C \to X$ be a continuous function. Then there exist an $x \in C$ such that

$$\|x - f(x)\| = d(f(x), C) = inf\{\|y - f(x)\| : y \in C\}. \qquad (K)$$

Theorem 1.2. Let C be a nonempty, closed, convex subset of a Banach space X such that the metric projection is upper semicontinuous. If $f:C \to X$ is a continuous function and $f(C)$ is relatively compact exists an $x \in C$ such that

$$\|x - f(x)\| = d(f(x), C) = inf\{\|y - f(x)\| : y \in C\}.$$

In this paper, we will prove two theorems to extends the (*) to a more broad equation $F: CxC \to R$ and some applications as corollaries. The proofs of the theorems are based on Ky Fan's fixed point theorem [2] and Himmelberg's fixed point theorem [4]. We list the theorems here.

Theorem 1.3. Let C be a nonempty, compact, convex subset of a normal linear space X and $F:C \to 2^C$ an *usc* multifunction with $F(x)$ nonempty, closed and convex for all $x \in C$, Then F has a fixed point

Theorem 1.4. Let C be a nonempty, closed and convex subset of a normed linear space X and $F:C \to 2^C$ an *usc* multifunction with $F(x)$ nonempty, closed, convex for all $x \in C$. If $F(C)$ is contained in a compact subset of C, then F has a fixed point.

2. In this section, we extend the Ky Fan's best approximation theorem.

Theorem 2.1. Let C be a nonempty, compact, convex subset of a normed linear space X and let $F:C \times C \to R$ be a function such that

(i) $F:C \times C \to R$ be is continuous;

(ii) for every $y \in C$, the subset $\{x \in C : F(x,y) = 0\}$ is nonempty and convex.

Then there exists an $x_0 \in C$ such that $F(x_0, x_0) = 0$.

Proof. Define $G: C \to 2^C$ as follows:

$$G(y) = \{x \in C : F(x,y) = 0\}, \text{ for every } y \in C.$$

Condition (ii) implies that $G(y)$ is nonempty and convex and condition (i) implies that $G(y)$ is closed. Next we need to prove that G is *usc*, i.e. we need to show that for any closed subset $A \subseteq C, \{y \in C : G(y) \cap A \neq \Phi\}$ is closed. In fact, if $\{y_n\} \subseteq \{y \in C : G(y) \cap A \neq \Phi\}$ such that $y_n \to u$, as $n \to \infty$, for some $u \in C$. Then for n= 1,2,3,..., there exists $x_n \in G(y_n) \cap A$ such that $F(x_n, y_n) = 0$. Since C is compact, so is A. There exists a subsequence $\{x_{n(i)}\}$ of $\{x_n\}$ such that $x_{n(i)} \to v$, for some $v \in A$, as $i \to \infty$. Using condition (i) again, we get $F(v,u) = \lim_{i \to \infty} F(x_{n(i)}, y_{n(i)}) = 0$. Then $v \in G(u)$, and since $v \in A$ this yields $v \in G(u) \cap A$. Hence $u \in \{y \in C : G(y) \cap A \neq \Phi\}$, which implies the set $\{y \in C : G(y) \cap A \neq \Phi\}$ is closed and G is *usc*.

By Ky Fan's fixed point theorem (Theorem 1.3), G has a fixed point x_0, *i.e.* $x_0 \in G(x_0)$. Then x_0 satisfies $F(x_0, x_0) = 0$.

Lemma 2.1. Let C be a nonempty subset of a normed linear space X and let $g: X \to R$ be a function defined by

$$g(x) = d(x,C) = inf\{\|y - x\| : y \in C\}, \text{ for every } x \in X.$$

Then g is continuous.

Proof. For any $x, y \in X$ and any given $\varepsilon > 0$, there exists $z \in C$ such that $g(x) = d(x,C) \geq \|z - x\| - \varepsilon$ Now

$$g(y) = d(y, C) \leq \|z - y\| \leq \|z - x\| + \|y - x\| \leq g(x) + \varepsilon + \|y - x\|$$

Combining the above inequalities, we have $g(y) - g(x) \leq \varepsilon + \|y - x\|$.

Similarly we can prove $g(x) - g(y) \leq \varepsilon + \|y - x\|$. From the last two inequalities we conclude $|g(y) - g(x)| \leq \varepsilon + \|y - x\|$ and since ε is arbitrary this yields

$$|g(y) - g(x)| \leq \|y - x\|.$$

Hence g is continuous.

If C is a nonempty, closed, convex subset of a Hilbert space H, then C is a Chebyshev set. That is $P_C : H \to C$ is well defined and for every x, $P_C(x)$ is a singleton and P_C has the well known property:

$$\|P_C(x) - P_C(y)\| \leq \|x - y\| \text{ for every } x, y \in H$$

If C is just a nonempty subset of a normed linear space X, then the above inequality does not hold. In fact, P_C may have no definition. But $d(.,C) = inf\{\|y - \cdot\| : y \in C\} : X \to R$ is well defined and from the proof of Lemma 2.1, $d(.,C)$ always satisfies the following inequality $|d(x,C) - d(y,C)| \leq \|x - y\|$, for every $x, y \in X$.

Lemma 2.2. Let C be a closed, convex and proximal subset of a normed linear space X. Let $P_C : X \to C$ be the metric projection on C. Then for every $x \in X$, $P_C(x)$ is nonempty, closed and convex subset of C.

Proof. Since C is a proximal subset, so $P_C(x)$ is nonempty. If $\{y_n\} \subseteq P_C(x)$ such that $y_n \to u$ as $n \to \infty$, for some $u \in C$ (C is closed). Then $\|u - x\| \leq \|u - y_n\| + \|y_n - x\| = \|u - y_n\| + d(x, C)$. Since $\|u - y_n\| \to 0$, *as* $n \to \infty$, this implies $\|u - x\| \leq d(x, C)$. But we always have $\|u - x\| \geq d(x, C)$. Hence $\|u - x\| = d(x, C)$, *i.e.* $u \in P_C(x)$. Next we will

prove that $P_C(x)$ is convex. Suppose $y_1, y_2, \dots, y_n \in P_C(x)$ and $0 \le \lambda_1, \lambda_2, \dots, \lambda_n \le 1$, such that $\sum_{i=1}^{n} \lambda_i = 1$. Let $u = \sum_{i=1}^{n} \lambda_i y_i \in C$ (C is convex). Then

$$\|u - x\| = \left\| \sum_{i=1}^{n} \lambda_i y_i - \sum_{i=1}^{n} \lambda_i x \right\|$$

$$\le \sum_{i=1}^{n} \lambda_i \|y_i - x\|$$

$$= \sum_{i=1}^{n} \lambda_i d(x, C)$$

$$= d(x, C).$$

Combining the above inequalities and noting $\|u - x\| \ge d(x, C)$, we get $\|u - x\| = d(x, C)$ *i.e.* $u \in P_C(x)$ .Hence $P_C(x)$ is convex.

Remark : Let C be a nonempty, compact, convex subset of a normed linear space X. Then C is closed, convex and proximal. Let $f: C \to X$ be a continuous function. Define $F: C \times C \to R$ by $F(x, y) = \|x - f(y)\| - d(f(y), C)$ for any $x, y \in C$.

Lemmas 2.1 and 2.2 ensure that F satisfies conditions (i) and (ii) of Theorem 2.1. Hence there exists an $x_0 \in C$ such that $F(x_0, x_0) = 0$, *i.e.* $\|x_0 - f(x_0)\| = d(f(x_0), C)$. This includes Ky Fan's best approximation theorem.

Corollary 2.1. Let C be a nonempty, compact and convex subset of a normed linear space X and let $g: C \to X$ be a continuous function such that

(a) $C \subseteq g(C)$;

(b) for any convex subset $A \subseteq C, g^{-1}(A)$ is convex.

Then for any continuous function $f: C \to X$, there exists an $x_0 \in C$. such that

$$\|g(x_0) - f(x_0)\| = d(f(x_0), C).$$

Proof. Define $F: C \times C \to R$ by

$$F(x,y) = \|g(x) - f(y)\| - d(f(y), C), \text{ for any } x, y \in C.$$

By Lemma 2.1, it can be seen that F is continuous. For any $y \in C$, we have

$$\{x \in C : F(x,y) = 0\}$$

$$= \{x \in C : \|g(x) - f(y)\| = d(f(y), C)\}$$

$$= g^{-1}(P_C(f(y))).$$

By using Lemma 2.2, the continuity of g and condition (b), we know that the set $\{x \in C : F(x,y) = 0\}$ is nonempty, closed and convex. Applying Theorem 2.1, there exists an $x_0 \in C$ such that $F(x_0, x_0) = 0$, i.e. $\|g(x_0) - f(x_0)\| = d(f(x_0), C)$.

Corollary can be modified as below.

Corollary 2.1.1. Let C be a nonempty, compact, convex subset of a normed linear space X and let $g: C \to X$ be a continuous function such that

(a) $\partial C \subseteq g(C)$;

(b) for any convex subset $A \subseteq \partial C$, $g^{-1}(A)$ is also convex,

Then for any continuous function $f: C \to X/C$, there exists $x_0 \in C$ such that

$$\|g(x_0) - f(x_0)\| = d(f(x_0), C).$$

It should be noted that any nonempty, closed, convex subset C of a Hilbert space H is a Chebyshev set. If $g: C \to X$ is a one to one mapping, then g will satisfy condition (b) in Corollary 2.1 So the following corollary should be a special case of Corollary 2.1.

Corollary 2.2. Let C be a nonempty, compact, convex subset of a Hilbert space H and let $g: C \to X$ be a continuous function such that

(a) $C \subseteq g(C)$;

(b) $g: C \to X$ is a one-to-one mapping.

Then for any continuous function $f: C \to H$, there exists an $x_0 \in C$ such that

$$\|g(x_0) - f(x_0)\| = d(f(x_0), C).$$

3. In this section we will give an extension of Reich's theorem (Theorem 1.2) which extends Theorem 2.1 by weakening the compact condition on C to closed condition.

Theorem 3.1. Let C be a nonempty, closed convex subset of a normed linear space X and let $F: C \times C \to R$ be a function such that

(i) $F: C \times C \to R$ is continuous;

(ii) for every $y \in C$, the subset $\{x \in C : F(x, y) = 0\}$ is nonempty and convex;

(iii) $\bigcup_{y \in C} \{x \in C : F(x, y) = 0\}$ is contained in a compact subset of C

Then there exists $x_0 \in C$ such that $F(x_0, x_0) = 0$.

Proof. Define $G: C \to 2^C$ as follows

$$G(y) = \{x \in C : F(x, y) = 0\}$$

Like the proof of Theorem 2.1, we can show that for every $y \in C, G(y)$ is nonempty, closed and convex and G is upper semicontinuous mapping. Now

$$G(C) = \underset{y \in C}{\cup} G(y) = \underset{y \in C}{\cup} \{x \in C : F(x, y) = 0\}$$

Condition (iii) implies that $G(C)$ is contained in a compact subset of C. From Himmelberg's fixed point theorem (Theorem 1.4), G has a fixed point x_0, *i.e.* $x_0 \in G(x_0)$. Then x_0 satisfies $F(x_0, x_0) = 0$.

Next Corollary is similar to Reich's Theorem.

Corollary 3.1. Let C be a nonempty, closed, convex and proximal subset of a normed linear space X such that for every compact subset A of $X, P_C(A)$ is also compact. If $f: C \to X$ is continuous and $f(C)$ is contained in a compact subset of C, then there exists an $x_0 \in C$ such that

$$\|x_0 - f(x_0)\| = d(f(x_0), C).$$

Proof. Define $F: C \times C \to R$ by

$$F(x, y) = \|x - f(y)\| - d(f(y), C)$$

Lemmas 2.1 and 2.2 and the proximal property of C assure that F satisfies conditions (i) and (ii) of Theorem 3.1. Since

$$\underset{y \in C}{\cup} \{x \in C : F(x, y) = 0\} = \underset{y \in C}{\cup} P_C(f(y)) = P_C(f(C)),$$

and $f(C)$ is contained in a compact subset of C and P_C maps

compact subset of X to compact subset of C, we get that $\bigcup_{y\in C}\{x\in C: F(x,y)=0\}$ is contained in a compact of C which satisfies condition (iii) for F. Then by Theorem 3.1., there exists an $x_0\in C$ such that $F(x_0,x_0)=0$, i.e. $\|x_0-f(x_0)\|=d(f(x_0),C)$.

Corollary 3.2. Let C be a nonempty, closed convex and proximal subset of a normed linear space X and let $g: C\to X$ be a continuous function such that

(a) $\partial C\subseteq g(C)$;

(b) for any convex subset $A\subseteq\partial C$; $g^{-1}(A)$ is also convex,

(c) $g^{-1}(\partial C)$ is contained in a compact subset of C.

Then for any continuous function $f: C\to X/C$, there exists $x_0\in C$ such that

$$\|g(x_0)-f(x_0)\|=d(f(x_0),C).$$

Proof. Define $F: C\times C\to R$ by

$$F(x,y)=\|g(x)-f(y)\|-d(f(y),C).$$

Lemma 2.1 implies that F is continuous. For any $y\in C$, we have

$$\begin{aligned}&\{x\in C: F(x,y)=0\}\\ &=\{x\in C: \|g(x)-f(y)\|=d(f(y),C)\}\\ &=g^{-1}(P_C(f(y))).\end{aligned}$$

By using the proximal property of P_C, Lemma 2.2, the continuity of g and condition (b), the set $\{x\in C: F(x,y)=0\}$ is nonempty, closed and convex. Now

$$\bigcup_{y\in C}\{x\in C: F(x,y)=0\}=\bigcup_{y\in C}g^{-1}P_C(f(y))\subseteq g^{-1}(\partial C).$$

This yields that $\bigcup_{y\in C} \{x \in C : F(x,y) = 0\}$ is contained in a compact subset of C. Hence F satisfies all the conditions of Theorem 3.1. Thus there exists $x_0 \in C$ such that $F(x_0,x_0) = 0$ that is

$$\|g(x_0) - f(x_0)\| = d(f(x_0), C).$$

The next corollary can be proved by combining the proofs of Corollaries 3.1 and 3.2 We state it without proof.

Corollary 3.3. Let be a nonempty, closed, convex and proximal subset of a normed linear space X such that for every compact subset A of $X, P_C(A)$ is also compact, Let $g : C \to X$ be a continuous function such that

(a) $C \subseteq g(C)$;

(b) for any convex subset $A \subseteq C, g^{-1}(A)$ is also convex.

If $f: C \to X$ is continuous and $f(C)$ is contained in a compact subset of X, then there exists an $x_0 \in C$ such that

$$\|g(x_0) - f(x_0)\| = d(f(x_0), C).$$

Remarks : (1) It can be seen that Corollary 2.1 is a special case of Corollary 3.3

(2) If C is a nonempty, closed convex subset of a Hilbert space H, then C is a Chebyshev set and for every $x, y \in H$, $\|P_C(x) - P_C(y)\| \le \|x - y\|$. Hence for every compact subset A of $H, P_C(A)$ is also compact. As a special case of Corollary 3.3, if $f : C \to H$ is continuous and $f(C)$ is contained in a compact subset of H, then there exists $x_0 \in C$ such that

$$\|g(x_0) - f(x_0)\| = d(f(x_0), C).$$

The following corollary will provide a property of the existence of eigenvectors and eigenvalues of some mappings in Hilbert spaces.

Corollary 3.4. Let B_r be a ball with radius $r > 0$ and centre, θ, the origin element of a normed linear space X. If $f: B_r \to B_r$ is continuous such that $f(B_r)$ is contained in a compact subset of B_r, then for any real $\lambda \geq sup\ \{\|f(y)\| : y \in B_r\}/r$, there exists an $x_0 \in B_r$, such that $f(x_0) = \lambda x_0$.

Proof. If $sup\ \{\|f(y)\| : y \in B_r\} = 0$, then $f(B_r) = \{\theta\}$. We can see that θ is a solution of the equation $f(x) = \lambda x$, for any real λ. So we may assume $sup\ \{\|f(y)\| : y \in B_r\} > 0$.

Define $F: B_r \times B_r \to R$ by

$$f(x, y) = \|\lambda x - f(y)\|, \text{ for any } x, y \in B_r.$$

From Lemma 2.1, F is continuous. For any $y \in B_r$, we have

$$\{x \in B_r : F(x, y) = 0\}$$

$$= \{x \in B_r : \lambda x = f(y)\}$$

$$= \{f(y)/\lambda\}.$$

Since for any $y \in B_r, \lambda \geq \|f(y)\|/r$, so $\|f(y)\|/\lambda \leq r$, this yields $f(y)/\lambda \in B_r$. Hence, since $\{x \in B_r : F(x, y) = 0\}$ is a singleton, it is nonempty, closed and convex, and we write

$$\bigcup_{y \in B_r,} \{x \in B_r : F(x, y) = 0\} = \bigcup_{y \in B_r,} \{f(y)/\lambda\} = f(B_r)/\lambda.$$

we have that $\bigcup_{y \in B_r} \{x \in B_r : F(x, y) = 0\}$ is contained in a compact subset of B_r because $f(B_r)$ is contained in a compact subset of B_r. By Theorem 3.1, there exists an $x_0 \in B_r$, such that $f(x_0) = \lambda x_0$.

4. In this section, we will concentrate on Hilbert spaces and Banach spaces. We will prove theorems similar to Theorems 2.1 and 2.2 without using Ky Fan's fixed point theorem and Himmelberg's fixed point theorem.

In a Hilbert space H, any nonempty, closed, convex subset C is a Chebyshev set. Hence, for such subset, $C, P_C : C \to H$ is a well defined singlevalued function.

Theorem 4.1. Let C be a nonempty, closed, bounded, convex subset of a Hilbert space H and let $F: C \times C \to R$ be a function such that

(i) $F: C \times C \to R$ is continuous;

(ii) for every $y \in C$, the subset $D(y) := \{x \in C : f(x,y) = 0\}$ is nonempty and convex;

(iii) the mappings $P_{D(.)} : C \to C$ is nonexpansive.

Then there exists $y_0 \in C$ such that $F(y_0, y_0) = 0$

Proof. Since $P_{D(.)} : C \to C$ is nonexpansive, from Browder's fixed point theorem, $P_{D(.)}$ has a fixed point $y_0 \in C$, i.e. $P_{D(y_0)}(y_0) = y_0$. Hence $y_0 \in D(y_0) = \{x \in C : F(x, y_0) = 0\}$.

A special case of Theorem 4.1 is the following Corollary.

Corollary 4.1. Let C be a nonempty, closed, bounded, convex subset of a Hilbert space H and let $F: C \times C \to R$ be a function such that

(i) $F: C \times C \to R$ is continuous;

(ii) for every $y \in C$, the equation $F(x,y) = 0$ has one and only

one solution, which is denoted by $g(y)$;

(iii) the mappings $g: C \to C$ is nonexpansive.

Then there exists $y_0 \in C$ such that $F(y_0, y_0) = 0$.

An immediately application of Corollary 4.1 is the Corollary below.

Corollary 4.2. Let C be a nonempty, closed, bounded, convex subset of a Hilbert space H. If $f: C \to H$ is nonexpansive, then there exists $x_0 \in C$ such that

$$\|x_0 - f(x_0)\| = d(f(x_0), C).$$

Proof. Define $F: C \times C \to R$ by

$$F(x,y) = \|x - f(y)\| - d(f(y), C), \text{ for any } x, y \in C.$$

Since C is a Chebyshev set, then the unique solution of the equation $F(x,y) = 0$ is $P_C f(y)$. Frorr the condition $f: C \to H$ is nonexpansive and since $P_C: H \to C$ is also nonexpansive this implies $P_C f: C \to C$ is nonexpansive. Then this Corollary follows Corollary 4.1 immediately.

In the Banach spaces case, we need stronger conditions as below. The proof of the theorem is similar to the proof of Theorem 4.1. We state it without proof.

Theorem 4.2. Let C be a nonempty, closed, bounded, convex and Chebyshev subset of a Banach space B and let $F: C \times C \to R$ be a function such that

(i) $F: C \times C \to R$ is continuous;

(ii) for every $y \in C$, the equation, $F(x,y) = 0$, has and only one solution, which is denoted by $g(y)$;

(iii) the mapping $g:C \to C$ is densifying.

Then there exists $y_0 \in C$ such that $F(y_0, y_0) = 0$.

REFERENCES

1. Cheney, E.W. Introduction to approximation theory, Mc-Graw-Hill, New York, (1966)

2. Fan, Ky Extensions of two fixed point theorems of F.E. Browder, Math Z., 112(1969), 234-240.

3. Lassonde, M. On the use of *KKM*-multifunctions. J. Math. Anal. Appl. 97(1983), 151-201.

4. Himmelberg, C.J. Fixed points of compact multifunctions. J. Math. Anal. Appl. 38(1972), 205-207.

5. Park, S. Fixed point theorems on compact convex sets in topological vector spaces, Contemp. Math. 72 (1988), 183-191.

6. Park, S. Best approximations, inward sets, and fixed points, Progress in Approx. Theory, Academic Press(1991) 1151-1158.

7. Park, S., Singh, S.P. and Waston, B. Some fixed point theorems for composites of acyclic maps. Proc.Amer. Math. Soc. 121(1994) 1151-1158.

8. Prolla, J.B: Fixed point theorems for set-valued mappings and existence of best approximants, Numer. Funct. Anal. Optimiz., 5(1982-83), 449-455.

9. Reich, S. Approximate selections, best approximations, fixed points, and invariant sets, J. Math Anal. Appl. 62(1978), 104-113.

10. Reich, S. Fixed points in locally convex spaces, Math Z. 125(1972), 17-31.

11. Sehgal, V.M. and Singh, S.P. A generalization to multifunctions of Fan's best approximation, Proc. Amer. Math. Soc. 102(1998), 534-537.

12. Sehgal, V.M. and Singh, S.P. A theorem on best approximation, Numer. Funct. Anal. Optimiz. 10(1989), 181-184.

13. Singer, L. Some remarks on approximative compactness, Rev. Roumaine Math. Pures Appl. 9(1964), 167-177.

14. Singh, S.P. and Waston, B. Proximity maps and fixed points, J. Approx. Theory 39(1983), 72-76.

15. Waters, C.W. some fixed point theorems for radial contractions, nonexpansive and setvalued mappings, Ph.D. Thesis, Univ. of Wyoming (1984).

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COMMON FIXED POINTS OF SET-VALUED MAPPINGS II

P. P. MURTHY

The purpose of the present paper is to study in continuation of the author of this paper with Singh and Singh ([11]). These theorems in this paper are indeed extension of the Gregus [06], Fisher and Sessa [5], Diviccaro, Fisher and Sessa [02], Murthy, Cho and Fisher [13] and many other results of set-valued mappings.

INTRODUCTION

Guay, Singh and Whitfield [03], Dubey and Singh [04] and Kaulgud and Pai [07] have obtained very nice fixed point theorems for set-valued mappings which was initially introduced by Nadler [10].

Let (X,d) be a metric space and $CB(X)$ be the class of non-empty closed bounded subsets of X . For any non-empty subsets A, B of X we define

$$D(A,B)=\inf\{d(a,b):a\in A,b\in B\},$$

$$H(A,B)=\max\{\sup\{D(a,B):a\in A\},\sup\{D(A,b):b\in B\}\}.$$

The space $CB(X)$ is a metric space with respect to the above defined distance function H (see Kuratowski [07] p. 214 and Berge [01] p. 126).

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Nadler [10] has defined the ***contraction mapping principle*** for set-valued mappings which follows :

A set-valued mapings $F: X \to CB(X)$ is said to be ***contraction*** if there exists a real number $\alpha, 0 \leq \alpha < 1$ such that $H(Fx, Fy) \leq \alpha d(x, y)$ for all $x, y \in X$.

Here we ask the reader to consalt the paper [11] for any special definitions.

DEFINITION 1.1 An orbit for a set-valued mapping $F: X \to CB(X)$ at a point x_0 is a sequence $\{x_n\}$ at x_0 is said to be ***regular*** if

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \text{ and } d(x_{n+1}, x_{n+2}) < H(Fx_n, Fx_{n+1}).$$

MAIN RESULTS :

In this section, suppose that ϕ be the class of all function $\varphi: [0, \infty) \to [0, \infty)$ satisfying the following conditions;

(i) φ is non-decreasing,

(ii) upper-semi continuous,

(iii) for each $t > 0$, $\varphi(t) < t$.

Now we need the following lemma of Matkowski ([12]) :

LEEMA 2.1 Suppose $\varphi: [0, \infty) \to [0, \infty)$, which is non-decreasing, upper-semi continuous and $\varphi(t) < t$ for all $t > 0$, Then $\lim_{n \to \infty} \varphi^n(t) = 0$ where φ^n is the composition of φ^n -times.

Now we are ready to prove our main theorems by using Lemma(2.1) :

THEOREM 2.2 Let S and T be mappings of a metric space X into $C(X)$ and let X be x_0 – jointly orbitally complete for $x_0 \in X$. Suppose that for each $r > 0$ and for all $x, y \in X$ satisfying :

$$H^r(Sx,Ty) \le \varphi(\alpha d^r(x,y)+\beta\ max\ \{D^r(x,Sx), D^r(y,Ty), \frac{1}{2}[D^r(x,Ty)$$

$$+D^r(y,Sx)]\}) \qquad (2.1)$$

where $\alpha, \beta \in [0,1)$ and $\alpha+\beta=1, \varphi \in \phi$.

Then S and T have a common fixed point in X.

PROOF : Let $x_0 \in X$ for any $x_1 \in Sx_0$, then by Definition of H, there exists a point $x_2 \in Tx_1$, such that $d(x_1,x_2) \le H(Sx_0,Tx_1)$. The choice of the sequence $\{x_n\}$ in X guarantees that $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$ *for* $n \in R^+$. Now, we claim that $d(x_1,x_2) \le d(x_0,x_1)$. For this now we suppose that $d(x_1,x_2) > d(x_0,x_1)$ and $\varepsilon = d(x_1,x_2)$. Then by using (2.1) it follows that

$$\varepsilon = d(x_1,x_2) \le H(Sx_0,Tx_1)$$

$$\le [\varphi(\alpha d^r(x_0,x_1)+\beta max\{D^r(x_0,Sx_0), D^r(x_1,Tx_1), 1/2[D^r(x_0,Tx_1)$$

$$+D^r(x_1,Sx_0)]\})]^{1/r}$$

$$\le [\varphi(\{\alpha+\beta\}\varepsilon^r)]^{1/r},$$

$$= [\varphi(\varepsilon^r)] < \varepsilon,$$

a contraction. Therefore $d(x_1,x_2) \le d(x_0,x_1)$ and

$$d^r(x_1,x_2) \le H^r(Sx_0,Tx_1)$$

$$\le \varphi\{\alpha d^r(x_0,x_1)+\beta max\{D^r(x_0,Sx_0), D^r(x_1,Tx_1), \frac{1}{2}[D^r(x_0,Tx_1)$$

$$+D^r(x_1,Sx_0)]\})$$

$$\le \varphi(\{\alpha+\beta\}d^r(x_0,x_1)),$$

$$= \varphi(d^r(x_0,x_1)).$$

similarly, we have

$$d^r(x_3,x_2) \le \varphi(d^r(x_2,x_1)) \le \varphi^2(d^r(x_0,x_1)).$$

Hence, we have for all $n \in R^+$ and $r > 0$,

$$d^r(x_n,x_{n+1}) \le \varphi^n(d^r(x_0,x_1)).$$

By lemma (2.1) it follows that $\lim_{n\to\infty} d^r(x_n,x_{n+1}) = 0$, that is

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0 \tag{2.3}$$

Now in order to show that $\{x_n\}$ is a Cauchy sequence, it is enough to prove that the sequence $\{x_{2n}\}$ is a Cauchy sequence. Suppose that $\{x_{2n}\}$ is not a Cauchy sequence, then there is an $\varepsilon > 0$ such that for a sequence of even integer $\{n(k)\}$ defined inductively with $n = (1) = 2 \, and \, n(k+1)$ is the smallest even integer greater than $n(k)$ such that

$$d(x_{n(k+1)}, x_{n(k)}) > \varepsilon \tag{2.4}$$

So that

$$d(x_{n(k+1)-2}, x_{n(k)}) \le \varepsilon \tag{2.5}$$

it follows that

$$\varepsilon < d(x_{n(k+1)},x_{n(k)}) \le d(x_{n(k+1)},x_{n(k+1)-1}) + d(x_{n(k+1)-1},x_{n(k+1)-2}) + d(x_{n(k+1)-2},x_{n(k)})$$

for $k = 1,2,3,\ldots$ Using (2.3) and (2.5) we obtain

$$\lim_{k\to\infty} d(x_{n(k+1)},x_{n(k)}) = \varepsilon. \tag{2.6}$$

By traingle inequality, we have

$$\left|d(x_{n(k+1)},x_{n(k)}) - d(x_{n(k)},x_{n(k+1)-1})\right| \le d(x_{n(k+1)},x_{n(k+1)-1}),$$

and $$\left|d(x_{n(k+1)-1},x_{n(k)+1}) - d(x_{n(k)+1},x_{n(k+1)})\right| \le d(x_{n(k+1)},x_{n(k+1)-1}).$$

$$\lim_{k\to\infty} d(x_{n(k)}, x_{n(k+1)-1}) = \lim_{k\to\infty} d(x_{n(k+1)-1}, x_{n(k)+1}) = \varepsilon. \quad (2.7)$$

$$d(x_{n(k+1)}, x_{n(k)}) \le d(x_{n(k+1)}, x_{n(k)+1}) + d(x_{n(k)}, x_{n(k)+1}),$$

$$\le H(Tx_{n(k+1)-1}, Sx_{n(k)}) + d(x_{n(k)+1}, x_{n(k)}), \quad (2.8)$$

and using (2.1), we have

$$H^r(Tx_{n(k+1)-1}, Sx_{n(k)}) \le \varphi(\alpha d^r(x_{n(k+1)-1}) + \beta max\{(D^r(x_{n(k+1)-1}, Sx_{n(k+1)-1}),$$

$$D^r(x_{n(k)}, Tx_{n(k)}), 1/2[D^r(x_{n(k+1)-1}, Sx_{n(k)}) + D^r(x_{n(k)}, Tx_{n(k+1)-1})]\}) \quad (2.9)$$

Using (2.5), (2.6), (2.8), (2.9) and upper semi-contiunity of φ and taking $k \to \infty$ it follows that

$$\varepsilon \le [\varphi(\alpha\varepsilon^r)]^{1/r} \le [\varphi(\varepsilon^r)]^{1/r} < \varepsilon$$

a contradiction. Therefore $\{x_{2n}\}$ is a Cauchy sequence in X and since X is x_0-jointly orbitally complete metric space, so that the sequence $\{x_n\}$ for each orbit at x_0 is convergent in X. Therefore there exists a point $\xi \in X$ such that $x_n \to \xi$. Then again using (2.1), we have

$$D^r(x_{2n-1}, T\xi) \le H^r(Sx_{2n-2}, T\xi)$$

$$\le [\varphi(\alpha d^r(x_{2n-2}, \xi) + \beta max\{D^r(x_{xn-2}, Sx_{2n-2}), D^r(\xi, T\xi), 1/2[D^r(x_{2n-2}, T\xi)$$

$$+D^r(\xi, Sx_{2n-2}))]\})]^{1/r} \quad (2.10)$$

Now taking $n \to \infty$ in (2.10), then we have

$$D^r(\xi, T\xi) \le \varphi(\beta D^r(\xi, T\xi)),$$

if $\xi \notin T\xi$ a contradiction. Thus $\xi \in T\xi$.

Similarly, we can show $\xi \in S\xi$. Hence, $\xi \in S\xi \cap T\xi$.

This completes the proof.

THEOREM 2.3 Let $S : X \to C(X)$ be a set-valued mappings satisfying for each $r > 0$ and all $x, y \in X$.

$$H^r(Sx, Sy) \leq \varphi(\alpha d^r(x, y) + \beta max\ \{D^r(x, Sx), D^r(y, Sy),$$

$$1/2[D^r(x, Sy) + D^r(y, Sx)]\}) \tag{2.11}$$

Where $\alpha, \beta \in (0, 1), \alpha + \beta = 1$ and $\varphi \in \phi$. Further, we assume that there exists a regular orbit $\{x_n\}$ for some $x_0 \in X$ which has a subsequence convergine to some point $\xi \in X$. Then S has a fixed point.

PROOF : Let $\{x_n\}$ be a subsequence of $\{x_n\}$ and convergine to a point ξ in X. Since X is regular so we have by Definition (1.1),

$$d(x_n, x_{n+1}) \leq H(Sx_{n-1}, Sx_n).$$

Then arguing as in Theorem 2.2, we can see that the sequence $\{x_n\}$ will be a Cauchy sequence and conveges to a point ξ in X.

$$d(\zeta, S\zeta) \leq d(\zeta, x_{n+1}) + D(x_{n+1}, S\zeta),$$

$$\leq d(\zeta, x_{n+1}) + D(x_{n+1}, S\zeta),$$

$$\leq d(\zeta, x_{n+1}) + [\varphi(\alpha d^r(x_n, \zeta) + \beta max\{D^r(x_n, Sx_n), D^r(\zeta, S\zeta),$$

$$1/2[D^r(x_n, S\zeta) + D^r(\zeta, Sx_n)]\}\})]^{1/r}.$$

On taking $n \to \infty$, we have

$$D(\zeta, S\zeta) \leq 0 + [\varphi(0 + \beta max\{0, D^r(\zeta, S\zeta), 1/2[D^r(\zeta, S\zeta) + 0]\}\})]^{1/r}.$$

$$\leq [\varphi(0 + \beta max\{D^r(\zeta, S\zeta)\}\})]^{1/r},$$

$$< D(\zeta, S\zeta)$$

a contradiction. Thus $D(\zeta, S\zeta) = 0$ *implies* $\zeta \in S\zeta$.

As an immediate consequence of the Theorem 2.2, we have the following corollaries :

COROLLARY 2.4 : Let *S* and *T* be the set-valued mappings as in Theorem 2.2 and satisfying the following inequality :

$$H^r(Sx, Ty) \leq \beta max\ \{d(x,y), D^r(x,Sx), D^r(y,Ty),\ 1/2[D^r(x,Ty) + D^r(y,Sx)]\} \tag{2.13}$$

for all $x, y \in X$, each $r > 0$ and $\beta \in (0,1)$.

Then *S* and *T* have a common fixed point in *X*.

PROOF : If we define $\varphi(t) = \beta max(t), \varphi \in \phi$.

COROLLARY 2.5 : Let *S* and *T* be the set-valued mappings as in Theorem 2.2 and satisfying the inequality :

$$H^r(Sx, Ty) \leq \alpha d(x,y) + \beta[D^r(x,Sx) + D^r(y,Ty)] + \gamma[D^r(x,Ty) + D^r(y,Sx)] \tag{2.14}$$

for all $x, y \in X$ where $\alpha, \beta, \gamma \in [0, 1)$ and $\alpha + 2\beta + 2\gamma < 1$.

Then *S* and *T* have a common fixed point in *X*.

Proof follows easily from theorem 2.2.

REFERENCES

1. C. Berge, Topological Spaces, The Mac Millan Company, New York (1963).

2. M.L. Diviccaro, B. Fisher and S. Sessa, Common fixed point theorem of Gregus type, Publ. Math. (Debercen), 34(1987), 83-87.

3. M.D. Guay, K.L. Singh and J.H.M. Whitefield, Common fixed points for set-valued mappings, Bull. Dela Acad. Pol. Des. Sci. vol.30 (11-12) (1982), 545-551.

4. L.S. Dubey and S.P. Singh, On multi-valued contraction mappings, Bull. Math. Soc. Sci. Math. RSR, 14(1970), 307-310.

5. B. Fisher and S.Sessa, On a fixed point theorem of Gregus, IJMMS 9(1986), 22-28.

6. M. Gregus, A fixed point theorem in Banach spaces, Bull. Un.Mat.Ital. 17(A) (1980), 193-198.

7. N. N. Kaulgud and D. V. Pai, Fixed point theorems for set-valued mappings, Nieuw. Archite Voor Wuskunde 23(1975), 49-66.

8. K. K. Kuratowski, Topology, vol. 1, Academic Press, New York (1966).

9. N. Negoescu, Observations sur des paircs D'applications multivoques D'un certain type De contractive, Bule.Inst.Politechnic Din IAST Tonmul 25 (3-4) (1989), 21-25

10. S. B. Nadler, Multi-valued contraction mappings, Pacific J. Math. 30(1969), 475-448.

11. M. R. Singh, L. S. Singh and P. P. Murthy, Common Fixed points of set-valued mappings, IJMMS vol.22 (1999).

12. J. Matkowski, Fixed point theorems for mappings with contractive iterate at a point, Proc. Amer. Math. Soc. 62(1977), 344-348.

13. P. P. Murthy, Y. J. Cho and B. Fisher, Common fixed points of Gregus type mappings, Glasnik Math. 30(50)(1995), 344-341.

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट, खण्ड १ (१९९८), अंक २, १५५-१६०
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A FIXED POINT THEOREM FOR A GENERALIZED NONLINEAR CONTRACTION

R.C. DIMRI* and U.C. GAIROLA**

In the present paper we will introduce the concept of R-weak commutativity for a pair of maps in probabilistic metric spaces and establish a fixed point theorem for generalized nonlinear contraction. Our result is a generalization of many well known previous results.

AMS(MOS) Subject Classifications : 47H10, 54H25

Key Words and Phrases : Fixed point, Probabilistic metric space, *R*-weak commutativity.

INTRODUCTION

Sehgal and Bharucha-Reid [8] initiated the study of fixed points in a sub class of probabilistic metric space. In view of the fact that probabilistic metric spaces render the concept of distance as a probabilistic rather than a deterministic one, Bharucha-Reid [1] underlines the importance of the study of fixed points in probabilistic spaces. Subsequently a number of fixed point theorem were proved in probabilistic metric and menger spaces (see, for instance, [2] [3], [4], [5], [6], [11], [12], [13] and their references). Motivated by a results of Chang Kim [2] and Sessa [9], Dimri [4] proved a result for generalized nonlinear contraction by introducing the concept of weakly commuting maps for probabilistic metric space. Recently Pant [7] gave an interesting concept of *R*-weak commutativity for a pair of

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map. Pant (op. cit) claims that *R*-weak commutativity is weaker than weak commutativity. Combining the results of Chang-Kim (op.cit) Dimri (op.cit) Pant [op.cit], we proved a fixed point theorem introducing the definition of *R*-weak commutativity for the pair of maps on probabitisic metric space.

Before coming to our main results the following definitions will be useful for further discussion.

Definition 1[4]. Two mappings *f* and *g* on on a probabilistic metric space *X* will be called a generalized nonlinear contraction pair (*f;g*) if and only if there exists a mapping $h: R \to R_+$ which is continuous from the right, $h(x) = 0$ if $x \leq 0$ and $0 < h(x) < x$ for $x > 0$ and satisfies

$$F_{fu,fv}(h(x)) \geq min\left\{F_{fu,gu}(x), F_{fv,gv}(x), F_{gu,gv}(x), F_{fv,gu}(x), F_{fu,gv}(x)\right\} \quad (1)$$

Definition 2. [4], [12]. Two self maps *f* and *g* on a probabilistic metric space X will be called weakly commuting if

$$F_{fgu,gfu}(x) \geq F_{fu,gu}(x)$$

for all $u \in X$ and $x > 0$

Definition 3[7]. Let (*M,d*) be a metric space and let *f* and *g* be self mappings of *M*. The mappings *f* and *g* will be called *R*-weakly commuting, provided there exists some positive real number *R* such that

$$d(fgx, gfx) \leq Rd\,(fx, gx)$$

for each *x* in *M*. *f* and *g* will be called *R*-weakly commuting at a point *x* if $d(fgx, gfx) \leq Rd(fx, gx)$ for some $R > 0$. Obviously, weak commutativity implies *R*-weak commutativity. However, *R*-weak commutative implies weak commutativity only when $R \leq 1$.

We now introduce the following :

Definition 4. Let *f* and *g* be two mappings on probabilistic metric space *X*. The pair (*f,g*) will be called *R*-weakly commuting if and only

if

$$F_{fgu,gfu}(Rx) \geq F_{gu,fu}(x) \text{ for all } u \in X \text{ and } R > 0.$$

The following definition is from [4].

Definition 5. Let *A* be a non empty subset of a probabilistic metric space *X*. The function

$D_A(x) = sup_{\in < x}\{inf_{u,v \in A} F_{u,v}(\in)\}$ is called the probabilistic diameter of *A*, the set *A* is said to be bounded if $sup_{x \in R} D_A(x) = 1$.

For each $u \in X$, let $O(f^n u)$ denote the sequence of iterates of *f*, that is, $O(f^n u) = \bigcup_{i=n}^{\infty}\{f^i u\}, n = 0,1,2,\ldots$.

The following is the main result of this paper.

Theorem 1: Let (X,F,t) be a menger space where *t* is continuous and satisfies $t(x,x) \geq x$ for every $x \in [0,1]$. Let (*f*;*g*) be a generalized nonlinear contraction pair of *R*-weakly commuting mapings satisfying the following conditions.

(a) There exists a sequence $\{u_n\}$ in *X* such that $fu_n = gu_{n+1}$, $n = 0,1,2,\ldots$ and $O(gu_n)$ is bounded;

(b) The sequence $\{gu_n\}$ has a subsequence converging to a point in $g(X)$.

Then f and g have a unique common fixed point and $\{gu_n\}$ converges to the fixed point.

Proof. For $i,\ j \geq n+1$, by (1)

$$Fgu_{i+1},\ gu_{j+1}(h(x)) = Ffu_i,\ fu_j(h(x))$$

$$\geq min\ \{Fgu_{i+1}, gu_i(x), fgu_{j+1}, gu_j(x)$$

$$Fgu_i, gu_j(x), Fgu_{j+1}, gu_i(x),\ Fgu_{i+1},\ gu_j(x)\}.$$

Since $gu_{i+1}, gu_{j+1}, gu_i, gu_j \in O(gu_n)$, we have on taking *inf* over all

$i, j \geq n+1$, $D_o(gu_n)\,(h(x)) \geq D_o(gu_{n-1})\,(x)$ for any $x > 0$.

Also, for $x > 0$, $\{D_0(gu_n)\,(x)\}$ is an increasing sequence. So for some $0 \leq P(x) \leq 1, D_0(gu_n)\,(x) \rightarrow P(x)$. we claim that $P(x) = 1$.

Suppose a $P(x_0) < 1$ for some $x > 0$. Then there exists $x > 0$ such that $D_0(gu_0)\,(x) > P(x_0)$ as $O(gu_n)$ is bounded. Let n be a positive integer with $h^n(x) \leq x_0$. Then $D_o(gu_n)\,(x_o) \geq D_o(gu_n)\,(x_0) \geq D_0(gu_n)\,(h^n(x)) \geq D_0(gu_0)\,(x) \geq P\,(x_0)$, a contradiction.

Hence $P(x) = 1$ for all $x > 0$, showing that $\{gu_n\}$ is a cauchy sequence and therefore converges to a point *p* in *g(x)*, so there exists a point *z* in *X* such that *gz* = *p*.

Let $U_{gz}(\in, \lambda)$ be any neighbourhood of *gz*. Since $gu_n \rightarrow gz$, for $\in, \lambda > 0$, there is an integer *N* such that

$$Fgu_{n+1}, gz^{(\varepsilon - h(\varepsilon))} > 1 - \lambda,\ Fgu_{n+1}, gu_n(\varepsilon) > 1 - \lambda, \text{ for } n \geq N. \tag{2}$$

Now by (1)

$$Ffz, u_{n+1}{}^{(h(\varepsilon))} = \quad Ffz, u_n{}^{(h(\varepsilon))} \geq min\Big\{Ffz, gz^{(\varepsilon)}, Fgu_{n+1}, gu_n{}^{(\varepsilon)},$$

$$Fgz, u_n{}^{(\varepsilon)},\ Fgu_{n+1}, gz^{(\varepsilon)},\ Ffz, gu_n{}^{(\varepsilon)}\Big\}$$

$$\geq min\Big\{Ffz, gzu_{n+1}^{(h(\varepsilon))},\ Fgu_{n+1}, gz^{(\varepsilon - h(\varepsilon))},$$

$$Fgu_{n+1}, gu_n^{(t)}, \mathrm{F}gz, gu_n^{(t)}\mathrm{Fgu}_{n+1}, \mathrm{gz}^{(t)},$$

$$Ffz, gu_{n+1}^{(h(\varepsilon))},\ Fgu_{n+1}, gu_n{}^{(\varepsilon - h(\varepsilon))}\Big\} > 1 - \lambda, \text{ by (2)}.$$

Thus

$$gz = fz. \tag{3}$$

Now for any $x > 0$, the *R*-Weak commutativity of *f* and *g* implies

$$Ffgz, gfz^{(RX)} \geq Fgz, fz^{(x)} = 1.$$

Therefore, from (3)

$$fgz = gfz = ggz = ffz. \tag{4}$$

Also

$$Fffz, fz^{(h(\varepsilon))} \geq min \left\{Ffz, gz^{(\varepsilon)}, Fffz, gfz^{(\varepsilon)}, Fffz, gz^{(\varepsilon)}, Ffz, gfz^{(\varepsilon)}\right\}$$

gives *ffz=fz* that is, *fz* is a fixed point of *f*. From (4) we conclude that *fz* is a fixed point of *g* also. Uniqueness of the common fixed point follows from (1).

Remark 1: By setting *g=1* (indentity mapping on *X*) in theorem 1 we get a stringthened form of the generalized contraction introduced by Ćirić[3] in *PM*-spaces. Moreover, this result is an improvement over the main result of Tiwari and Pant [13] which has been proved taking *g* continuous.

Now the following result, being the metric analogue of Theorem 1 is superior to the result of Sessa [9] and Singh [10] (see also the reference of [10]. It is to be noted that Sessa [9] has taken both *f* and *g* continuous whereas Singh [10] has taken them commuting.

Theorem 2. Let *f* and *g* be a pair of *R*-weakly commuting self mapping of a metric space (M,d) satisfying the following conditions :

(i) there is a mapping $h: R_+ \to R_+$ which is continuous from the right $h(x) = 0$ if $x \leq 0$ and $0 < h(x) < x$ for $x > 0$ and satisfies $d(fu, fv) \leq h \left\{(d(fv, gv), d(gu, gv), d(fv, gu), d(fu, gv)\right\}$ for all $u, v \in M$;

(ii) there exists a sequence $\{u_n\}$ in *M* such that $gu_{n+1} = fu_n, n = 0,1,...$ and $sup\left\{d(gu_i, gu_j), i, j = 0,1,2,...\right\} < \infty$;

(iii) the sequence $\{gu_n\}$ has a subsequence converging to a point in $g(M)$.

Then f and g have a unique common fixed point and converges to the fixed point.

Proof . Noting that the metric space (M,d) is a *PM*-space with $F_{u,v}(x) = H(x - d(u - v))$, where

$$H(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$

the proof follows from [3].

REFERENCES

1. A.T. Bharucha-Reid, Fixed point theorems in probabilistic analysis, Bull. Amer Math, Soc, 82(1976), 641-657.
2. J.I. Chang and C.W. Kim,Fixed points of generalized contraction mappings on Menger spaces. J. Korean. Math, Soc. 19(2) (1983). 135-141.
3. Lj. B Ćirić, On fixed points of generalized contraction on probabilistic metric spaces, Publ. Inst. Math. (Beograd) (NS) 18(1975), 71-78.
4. R.C. Dimri, Some fixed point theorems in Menger spaces, D. Phill thesis, H.N.B. Garhwal University Srinagar Garhwal, 1988.
5. R.J. Eghert, Products and gualients of probabilistic metric spaces, Pacific J. Math. 24(1968), 437-455.
6. O. Hadžic, Common fixed point theorems for family of mapping in complete metric spaces, Math, Japon. 29(1984), 127-134.
7. R.P. Pant, Common fixed points of noncommuting mapping, J. Math. Anal. Appl. 188(2), (1994) 436-440.
8. V.M. Sehgal and A.T. Bharucha-Reid, Fixed points of contraction mapping on probabilistic metric spaces, Math. Systems Theory 6(1972), 97-102.
9. S. Sessa, On a weak commutativity condition of mapping in fixed point consideration, Publ. Inst. Math. (Beograd) (N.S). 32(46) (1982), 149-153.
10. S.L. Singh, A note on a recent fixed point theorems for commuting mappings, Vijñāna Parishad. Anusandhan Patrika 26(1983), 259-261.
11. S.L. Singh, S.N. Mishra and B.D. Pant, General fixed point theorems in probabilistic metric and uniform spaces, Indian J. Math. 29(1987), 9-21.
12. S.L. Singh and B.D. Pant, Coincidence and fixed point theorems for a family fo mappings on Menger spaces and extension to uniform spaces, Math. Japon. 33(6) (1988), 957-973.
13. B.M.L. Tiwari and B,D. Pant, Fixed points of a pair of mappings in probabilsitic metric spaces, Jnānābhā 13(1983), 13-25.

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट, खण्ड १ (१९९८), अंक २, १६१-१७०
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FIXED POINT THEOREMS FOR JUNGCK TYPE HYBRID CONTRACTIONS*

CHITRA KULSHRESTHA**

The purpose of this paper is to show the existence of a common fixed point for triplets of maps (P,Q,f) and (T,f,g) satisfying

$$\delta(Px,Qy) \leq q \max\{d(fx,fy),H(fx,Px),H(fy,Qy),\tfrac{1}{2}[H(fx,Qy)+H(fy,Px)]\}$$

and

$$\delta(Tx,Ty) \leq q \max\{d(fx,gy),H(fx,Tx),H(gy,Ty),\tfrac{1}{2}[H(fx,Ty)+H(gy,Tx)]\}$$

for all x,y in X, wherein (X,d) is a metric space, H is the Hausdorff metric induced by d, $\delta(A,B):=\sup\{d(x,y):x\in A, y\in B\}, P,Q,T: X\to CB(X)$, $f,g: X\to X, 0<q<1$ and $CB(X)$ is the collection of all nonempty closed and bounded subsets of X.

INTRODUCTION

Following the Banach contraction mapping, Nadler (1969) introduced the concept of multi-valued contraction mappings and established that a multi-valued contraction mapping possesses a fixed point in a complete metric space. Subsequently a number of fixed-

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point theorems in metric spaces have been proved for multi-valued mappings satisfying contractive conditions.

Jungck (1976) generalised the Banach contraction principle by introducing a contractive condition for a pair of commuting mappings in a metric space. In this paper we are establishing fixed point theorems for

(i) two multi-valued mappings, each commuting with a single-valued mapping and

(ii) two single valued mappings, each commuting with a multi-valued mapping.

PRELIMINARIES

Let (X,d) be a metric space. We shall follow the following notations and definitions :

$CL(X) = \{A : A$ is a nonempty closed subset of $X\}$,

$CB(X) = \{A : A$ is a nonempty closed and bounded subset of $X\}$,

$D(A,B) = \inf\{d(a,b) : a \in A, b \in B, A, B \in CL(X)\}$,

$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B, A, B \in CL(X)\}$,

$N(\varepsilon, A) = \{x \in X : d(x,a) < \varepsilon$ for some $a \in A, \varepsilon > 0, A \in CL(X)\}$,

$$\text{and } H(A,B) = \begin{cases} \inf \varepsilon > 0 : A \subseteq N(\varepsilon, B) \text{ and } B \subseteq N(\varepsilon, A), A, B \in CL(X), \\ \qquad\qquad \text{if the infimum exists} \\ \infty \qquad\qquad \text{otherwise.} \end{cases}$$

H is called the generalized Hausdorff distance function for $CL(X)$ indced by d. If $H(A,B)$ is defined for $A, B \in CB(X)$ then the pair

(X,H) is a metric space and H is called the Hausdroff metric induced by d. $D(x,A)$ will denote the ordinary distance between $x \in X$ and A, a nonempty subset of X. Let f be a single-valued mapping from X to X and T a multi-valued mapping from X to the nonempty subsets of X.

Definition : (Itoh and Takahashi, 1977) A multi-valued mapping $T: X \to CL(X)$ and a single-valued mapping $f: X \to X$ are said to commute if for each x in X

$$fTx = f(Tx) \subseteq T(fx) = Tfx.$$

Lemma : (Rus, 1975) Let $A \in CB(X)$ and $0 < \alpha < 1$ be given.

Then for every $x \in X$, there exists an $a \in A$ such that

$$d(x,a) \geq \alpha\,\delta(x,A) \text{ and } d(x,a) \geq \alpha\, H(x,A).$$

MAIN RESULTS

Theorem 1: Let (X,d) be a complete metric space and P, Q be two multi-valued mappings from X to $CB(X)$. If there exists a mapping $f: X \to X$ commuting with each of P and Q, such that $P(X) \cup Q(X) \subseteq f(X)$ and

(1) $\delta(Px,Qy) \leq q.\max\ \{d(fx,fy), \delta(fx,Px), \delta(fy,Qy), ½[\delta(fx,Qy)+\delta(fy,Px)]\}$ for every x, y in X, q in (0, 1). Then P, Q and f have a common fixed point, i.e., there exists a z in X such that $Pz = Qz = \{fz\} = \{z\}$.

Proof : Let $k = q^a$ for some a in (0, 1), k is a positive number less than 1. Let us define single-valued mappings P_1 and Q_1 each commuting with f, from X into itself such that $P_1x \in Px$, $Q_1x \in Qx$ for all x in X, and

$d(fx, P_1x) \geq k.\delta(fx, Px)$ with $P_1 f = fP_1$

and $d(fx, Q_1x) \geq k.\delta(fx, Qx)$ with $Q_1 f = fQ_1$.

Lemma together with the commutativity of f with each of P and Q justifies the choice of P_1 and Q_1.

Then we get by (1),

$$d(P_1x, Q_1y) \leq \delta(Px, Qy)$$

$$\leq q.\max\{d(fx, fy), \delta(fx, Px), \delta(fy, Qy), \tfrac{1}{2}[\delta(fx, Qy) + \delta(fy, Px)]\}$$

$$\leq q.\max\{d(fx, fy), k^{-1}d(fx, P_1x), k^{-1}d(fy, Q_1y), \tfrac{1}{2}k^{-1}[d(fx, Q_1y) + d(fy, P_1x)]\}$$

$$\leq q.\max\{d(fx, fy), q^{-a}d(fx, P_1x), q^{-a}d(fy, Q_1y), \tfrac{1}{2}q^{-a}[d(fx, Q_1y) + d(fy, P_1x)]\}$$

$$\leq q^{1-a}.\max\{q^a d(fx, fy), d(fx, P_1x), d(fy, Q_1y), \tfrac{1}{2}[d(fx, Q_1y) + d(fy, P_1x)]\}$$

$$\leq q^{1-a}.\max\{d(fx, fy), d(fx, P_1x), d(fy, Q_1y), \tfrac{1}{2}[d(fx, Q_1y) + d(fy, P_1x)]\}.$$

As X is complete it follows from Singh and Singh (1980) that P_1, Q_1 and f have a unique common fixed point, say, z in X, i.e. $P_1z = Q_1z = fz = z$.

Then $0 = d(z, P_1z) \geq k.\delta(\{z\}, Pz)$ implies $\delta(\{z\}, Pz) = 0$ giving $Pz = \{z\}$.

Similarly we can show that $Qz = \{z\}$. Hence $Pz = Qz = \{fz\} = \{z\}$, i.e. P, Q and f have a common point in X.

Theorem 2 : Let (X, d) be a complete metric space and P, Q be two multi-valued mappings from X to $CB(X)$. If there exists a

mapping $f:X \to X$ commuting with each of P and Q such that $P(X) \cup Q(X) \subseteq f(X)$ and

(2) $\delta(Px, Qy) \le q.\max\ \{d(fx, fy), H(fx, Px), H(fy, Qy), \frac{1}{2}[H(fx, Qy) + H(fy, Px)]\}$ for all x, y in X and some q in (0, 1), then P, Q and f have a common fixed point in X.

Proof : The proof can be completed exactly on the lines of Theorem 1 in view of the Lemma.

Theorem 3 : Let (X, d) be a complete metric space and P, Q be two set-valued mappings from X to $CB(X)$. If there exists a mapping $f:X \to X$ commuting with each of P and Q such that $P(X) \cup Q(X) \subseteq f(X)$ and for all x, y in X and some q in (0, 1)

(3) $\delta(Px, Qy) \le q.\max\ \{d(fx, fy), \delta(fx, Px), \delta(fy, Qy), \frac{1}{2}[D(fx, Qy) + D(fy, Px)]\}$

then P, Q and f have a common fixed point in X, i.e., there exists a z in X such that $Pz = Qz = \{fz\} = \{z\}$.

Proof : Let $k = q^a$ for some a in (0,1), k is a positive number less 1. Let us define single-valued mappings P_1 and Q_1 each commuting with f, from X into itself such that $P_1x \in Px$, $Q_1x \in Qx$ for all x in X, and

$$d(fx, P_1x) \ge k.\delta(fx, Px) \text{ with } P_1f = fP_1$$

and $$d(fx, Q_1x) \ge k.\delta(fx, Qx) \text{ with } Q_1f = fQ_1.$$

We have by (3),

$d(P_1x, Q_1y) \le \delta(Px, Qy)$

$\leq q.\max\{d(fx,fy),\delta(fx,Px),\delta(fy,Qy),\tfrac{1}{2}[D(fx,Qy)+D(fy,Px)]\}$

$\leq q.\max\{d(fx,fy),k^{-1}d(fx,P_1x),k^{-1}d(fy,Q_1y),\tfrac{1}{2}[d(fx,Q_1y)+d(fy,P_1x)]\}$

$\leq q.\max\{d(fx,fy),q^{-a}d(fx,P_1x),q^{-a}d(fy,Q_1y),\tfrac{1}{2}[d(fx,Q_1y)+d(fy,P_1x)]\}$

$\leq q^{1-a}.\max\{q^a d(fx,fy),d(fx,P_1x),d(fy,Q_1y),\tfrac{1}{2}q^a[d(fx,Q_1y)+d(fy,P_1x)]\}$

$\leq q^{1-a}.\max\{d(fx,fy),d(fx,P_1x),d(fy,Q_1y),\tfrac{1}{2}[d(fx,Q_1y)+d(fy,P_1x)]\}$.

Therefore, X being complete it follows that P_1, Q_1 and f have a common fixed point, say z in X, i.e. $P_1z=Q_1z=fz=z$.

Then $0=d(z,P_1z)\geq k.\delta(\{z\},Pz)$ implies $\delta(\{z\},Pz)=0$ giving $Pz=\{z\}$.

Similarly we can show that $Qz=\{z\}$.

Hence $Pz=Qz=\{fz\}=\{z\}$, i.e., P, Q and f have a common fixed point in X.

Our results generalize many results for two and three mappings under contractive conditions discussed by Jungck (1976), Kasahara (1978), Ranganathan (1978), Singh (1977) and Yeh (1979).

Theorem 4 : Let (X,d) be a complete metric space $f,g:X\to X$ and $T:X\to CB(X)$, such that $T(X)\subseteq f(X)\cap g(X)$. If f and g be continuous, T commuting with each of f and g and there exists q in (0, 1), such that

(4) $\delta(Tx,Ty)\leq q.\max\{d(fx,gy),H(fx,Tx),H(gy,Ty),\tfrac{1}{2}[H(fx,Ty)+H(gy,Tx)]\}$

for all x,y *in* X, then T,f,g have a unique common fixed point, i.e..

there exists a unique z in X such that $\{fz\} = \{gz\} = \{z\} = Tz$.

Proof : For some a in (0,1), let $k = q^a$.. Then k is a positive number less 1. Choose a single-valued mappings $h: X \to X$ such that for all x, y in X,

$$hx \in Tx, d(fx, hx) \geq k.H(fx, Tx) \text{ with } hf = fh$$

and $$hy \in Ty,\ d(gy, hy) \geq k.H(gy, Ty) \text{ with } hg = gh.$$

The Lemma together with the commutativity of T with each of f and g justifies the choice of h.

Then by (4), we have

$$d(hx, hy) \leq \delta(Tx, Ty)$$

$$\leq q.\max \{d(fx, gy), H(fx, Tx), H(gy, Ty), \tfrac{1}{2}[H(fx, Ty) + H(gy, Tx)]\}$$

$$\leq q.q^{-a}.\max \{kd(fx, gy), kH(fx, Tx), kH(gy, Ty), \tfrac{1}{2}[kH(fx, Ty) + k.H(gy, Tx)]\}$$

$$\leq q^{1-a}.\max \{d(fx, gy), d(fx, hx), d(gy, hy), \tfrac{1}{2}[d(fx, hy) + d(gy, hx)]\}.$$

For $k < 1$ this gives $kd(fx, gy) < d(fx, gy)$.

Since $q^{1-a} < 1$ and

$$h(X) = \bigcup_{x \in X} hx \subseteq \bigcup_{x \in X} Tx = T(X) \subseteq f(X) \cap g(X),$$

it follows from Singh (1979) that f, g and h have a unique common fixed point say z in X.

For the point z,

$$0 = d(fz, hz) \geq kH(\{fz\}, Tz) \qquad \text{(By Lemma)}$$

$$\geq kH(\{z\}, Tz)$$

Hence $Tz = \{z\}$.

To establish the uniqueness of z, let there be another point w such that $Tw = \{w\} = \{fw\} = \{gw\}$. Then by (4) and Lemma

$d(z,w) \leq \delta\,(Tz, Tw)$

$\leq q.\max\{d(fz, gw), H(fz, Tz), H(gw, Tw), \frac{1}{2}[H(fz, Tw) + H(gw, Tz)]\}$

$\leq q.\max\{d(z,w), 0, 0, \frac{1}{2}[d(z,w) + d(w,z)]\}$

giving $d(z,w) \leq q.d(z,w)$, which implies $z = w$. Hence the fixed point is unique.

Corollary : Let (X,d) be a complete metric space. If $T: X \to CB(X)$ is a multi-valued maping which satisfies

$\delta\,(Tx, Ty) \leq q.$ max $\{d(x,y), H(x, Tx), H(y, Ty), \frac{1}{2}[H(y, Tx) + H(x, Ty)]\}$

for all x, y in X and some q in (0, 1), then T has a unique fixed point z and $Tz = \{z\}$.

Proof : Taking $fx = gx = x$ for every x in X, the proof follows directly from Theorem 4. This corollary presents an improved version of Theorem 2 of Iseki (1974) under a slightly different condition.

Theorem 5: Let (X,d) be a complete metric space, $f, g: X \to X$ and

$T: X \to BN(X)$, a set of nonempty bounded subsets of X, such that $T(X) \subseteq f(X) \cap g(X)$. If f and g be continuous, T commuting with each of f and g, and there exists a q in (0,1) such that

(5) $\delta(Tx, Ty) \leq$

$$q.\max\{d(fx, gy), \delta(fx, Tx), \delta(gy, Ty), \tfrac{1}{2}[D(fx, Ty) + D(gy, Tx)]\}$$

for all x, y in X, then T, f and g have a unique common fixed point, i.e., there exists a unique z in X such that $\{fz\} = \{gz\} = \{z\} = Tz$

Proof : The proof is similar to that of Theorem 4.

REFERENCES

1. Iseki, K. (1974), Multivalued contraction mappings in complete metric spaces, Math, Sem. Notes Kobe Univ. 2, 45-51.

2. Itoh, S. and Takahashi, W. (1977), Single-valued mappings, multi-valued mappings and fixed point theorems, J. Math. Anal. Appl. 59(3), 514-521.

3. Jungck, G.(1976), Commuting mappings and fixed points, Amer, Math. Monthly 83(4), 261-263.

4. Kasahara, S. (1978), On some recent results on fixed points, Math. Sem. Notes kobe Univ. 6, 373-382.

5. Naddler, S.B. (1969) Jr. Multi-valued contraction mappings. Pacific J. Math, 30, 475-488.

6. Ranganathan S. (1978), A fixed point theorem for commuting mappings, Math. Sem. Notes Kobe Univ. 6, 351-357.

7. Rus I.A. (1975), Fixed point theorems for multi-valued mappings, Math. Japonica 20 : 21-24.

8. Singh, S.L. (1977), On common fixed points of commuting mappings, Math. Sem. Notes Kobe Univ. 5; 131-134.

9. Singh, S.L. (1979), A common fixed point theorem in 2-metric space, proc. Nat. Acad, Sci, India Sect A 55, pp 32.

10. Singh S.L. and Singh, S.P. (1980), A fixed point theorem, Indian J. Pure Appl. Math, 11, 1584-1586.

11. Yeh, C.C. (1979), On common fixed point theorems of continuous mappings, Indian J. Pure, Appl. Math. 10 : 415-420.

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INDIAN MATHEMATICS : A BRIEF HISTORIC EVOLUTION

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This paper presents a brief historical account of chronology of Vedic literature, elementary treatment of surd numbers, theorem of square on the diagonal, suqare- and cube-rooting, some aspects on varga *prakrti*, values of π, and tangent- and π- series. Datta's semi-geometrical approach for obtaining $\sqrt{2}$ is an ingenious technique based on the lines with the prescription of the *śulba* geometry. This method paves for evaluation of this and other surd numbers in various ways, thereby giving rise to a number of series representation for $\sqrt{N}$ ($N > 0$, a positive integer):

1. CHRONOLOGY

1.1 B.G. Siddharth, a noted historian of Hindu astronomy, presents the widely accepted theory on Indo Aryan in the following attractive words (cf. Calendric astronomy, astronomical dating and archaeology : A new view of antiquity and its science, B.M. Birla Science Research Communication, July 1993, p.1) : "According to generally accepted ideas, civilization and science begain in Egypt and Sumeria in the third/fourth mellenium B.C. and spread in various directions. In the context of the Indian subcontinent (that is South Asia), it is believed that Indo Aryans, an Indo European people,

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invaded the north western parts of the country somewhere around 1500 B.C., over running in the process the then existing Indus valley or Harappan civilization. The Indo Aryans, so the theory goes, were a semi-nomadic, hardy, rustic and illetrate lot who could overcome the civilized and settled Harappan inhabitants, destroying their dwellings in the process because of their superior strength and equistrian skill." He further adds [op. cit] : "This scenario is based on an interpretation of the earliest extant Indo European text, the Rig ("Rg") Veda. The *Rig Vedic* hymns are supposed to be invocations to various tribal/naturalistic dieties, for aid in their battles to conquer the original inhabitants." However contrary to this, along the lines with several indologists/historians criticized the theory in the recent years and in the past, Siddharth [op. cit] points out that "*Rig Veda* and the related Vedic literature on the contrary contain amazingly accurate and sophisticated claendric astronomy", adding "this infact points to not an illetrate semi-nomadic tribal society but rather a well settled agrarian and meticulously scholarly people. Once the astronomical content of the Vedic literature is recognized, several dates begin to emerge which *blatantly* contradict the prevalent picture of pre-history. It will then be shown that very recent archaeological excavations spanning a period of nearly six thousand years, from about 7500 B.C. upto about 1500 B.C. can be meaningfully understood against this background." Siddarth and others view in this regard is further strengthened once we go in details in recent archaeological findings (see, for instance, Aryans were not invaders from central Asia : Archaeologists debunk earlier theories, Hindustan Times Delhi Edn.) 2nd Sept. 1998, p.1. This is the excavation report on Kunal village in Hissar district of Haryana).

1.2 Chronology is the backbone of history of any subject or culture. Unfortunately, the absence of proper Indian chronology is creating serious confusions and controversies especially in dealing

with pre-historic and ancient periods. The core factor behind the lacking of proper chronology, what we visualize these days has arisen because whatever Indian writers wrote they generally dedicated to GOD and did not want credit by mentioning authorships, dates of compositions, etc. and more so in some writings they attributed as if the things (matters) were *enunciated* in antiquity. Here is M.S. Rangachari's view (J.I.I.Sc., Special Issue 1987, p.3) on Indian chronology :

"Before the advent of the westerner into the Indian soil our forefathers were indifferent to dating of any work. In most of the cases they were indifferent to the authorship too. The reason was either a philosophical attitude that the *fact were* just discovered by GOD's will or that the utility of the fact was more important. The dating suggested by western scholars for many of the works of the east is at best a conjecture."

To deal with such problems effectively a Chronology Committee was constituted in 1950 during a Symposium on History of Sciences in South Asia organised under the patronage of INSA (the then NISI). The committee constituted of well-known historians of scientific temprament, viz., A.S. Attekar, P.C. Bagchi, S.L. Hora, D.S. Kothari, A.N. Singh as members and R.C. Majumdar as Chairman and recommended the following working table (cf. Ganita Bharati 12 (1990), 17-26) :

Age of *Ṛg Veda*	2000-1500 B.C.
Age of Samhitas and Brahmanas	1500-800 B.C.
Age of old Upnisads	900-500 B.C.
Vedanga Jyotisa (present tent)	500 B.C.
Śulbasūtras	500 B.C. and later

Dharmasūtras	600-200 B.C.
Mahābhārata; also Manusmrti and *Ramayaṇa*	200 B.C.-200 A.D.

Till date, it could not serve the purpose perhaps one of the foremost reasons might be that subsequent writers were not aware of recommendations of such Committee. Gupta [op. cit.] rather advocates for a permanent joint Chronology Committee to be appointed by national bodies like National Commission on History of Science under INSA and Indian Council for Historical Research which should form some norms and examine the matter carefully and suggest a longer unambiguous working table from time to time which the researchers and others should follow. However those differing from suggested dates should present sufficient grounds to support their views. It should be reminded that a race of claiming fantastic early chronology be discouraged and utmost care be taken to decide controversial dates and even in some cases, whenever possible, confirmation of evidence may be sought from sources such as archaeology and epigraphy.

Somewhat more acceptable dates for Vedas and subsequent literature have been suggested by Satya Prakash and Sharma [65, pp. 5-6] as is evident from the para :

"If we accept 3000 B.C. as a convenient date for the *Rgvedic* culture, the *Aitareya Brahmaṇa* will have to be assigned a date 2500-2000 B.C., the *Satapatha Brahmaṇa* 1500 B.C., *Tattriya Samhita* 1600 B.C., Baudhāyana (*Śulba*) *Sūtras* (BSS) 800 B.C., Āpastamba (*Śulba*) *Sūtras* (ASS) 600 B.C., *Paṇiṇi Sūtras* 500 B.C. and the other *sūtras* would find a place in their respective order, retaining Katyāyāna (Śulba) Sūtras to (KSS) 200 B.C."

SURD NUMBERES

2.1 India has a rich and glorious mathematical tradition from Vedic period onwards till late medieval period (c. 1200-1800 A.D.), where a series of eminent scholars contributed much to the origin and development of Mathematics confined mainly to the fields : Algebra, Astronomy, Calculus, Geometry, Mensuration, Number Ssytem and Trigonometry (see [1], [3], [5], [6], [13], [18], [23A] and [34)]. Mathematics was held in high esteem as is clear from the dictum.

गणितं मूर्धनि स्थितम्

The very concept of irrationality of numbers in human civilization is a very important milestone in the history of science. It would not be amiss to include a brief development of square-root, cube-root of positive numbers and π.

2.2 BRIEF HISTORICAL DEVELOPMENT OF SQUARE- AND CUBE-ROOTS

Āryabhaṭa I (c. 499 A.d.) gave a method of extraction of square-root and cube-root (see [3], [6] and [18]) developed on the basis of decimal system. Mahāvīra (c. 850 A.D.), Śridhara (c. 850-950 A.D.) *Āryabhaṭa* II (c. 950 A.D.), *Bhāskara* II (c. 1150 A.D.) *Kamlākara* (c. 1658 A.D.) etc. have given the same/and similar methods of extracting suqare-root of a positive integer. [78]. Not all of them hae given cube-root methods. Brahmagupta (c. 628 A.D.), *Mahāvīra*, *Śridhara*, *Āryabhaṭa* II, *Bhāskara* II (see [3, p. 80]) have attempted to give methods for obtaining cube-roots (see also [2], [11], [12], [18], [39], [40], [45], [46], [53], [62], [71]-[73], [77] and [78]). Square-root and cube-root methods available in Hindu and Jaina works are found in Arabian works from 9th century and onwards [79, pp, 138-139]. In the sixteenth century A.D., both square-root and cube-

root methods were given by Cantaneo, which are exactly the same as those of Āryabhaṭa I [79, p. 148] it seems that the indian mathods of root extraction travelled to Europe via Arab and returned back from Europe to India in a cosmetized form.

2.3 BSS (1.62) gives the value of $\sqrt{2}$ in the following hemistich (see [5]) :

प्रमाणं तृतीयेन वर्धयेत् तच्च चतुर्थेन आत्मचतुर्स्त्रिशोनेन।

That is,

$$\sqrt{2} = 1 + 1/3 + 1/(3\text{x}4) - 1/(3\text{x}4\text{x}34) \qquad \text{(approx)}$$

$$= 557/408 = 1.4142157 \text{ (correct to 5th decimal places)}$$

The same rule is quoted in ASS (1.6) and KSS (2.5) (see [5]). Dutta [13] and Thibaut [82] gave possible methods of the solution to arrive at the value. Rodet [60] gave the solution by the method of successive approximation as

(A) $\sqrt{\pi} = \sqrt{(a^2 + r)} = a + r/(2a+1) + e_3 + e_4 + \ldots\ldots\ldots$

Where e_i is the i^{th} term approximation and a, the greatest root of $\sqrt{N}$

The methods described later on by Gurjar [23] and Gupta [24] are the same and no improvement over Rodet (see, for details [51]). See also [8, pp. 31-34], [10, p. 147], [11], [14], [15], [19, pp. 347 & 350]; [25], [37, pp. 51 & 354], [47, p. 343], [49], [51, p. 78], [66] and [79, pp. 236, 254 & 259].

The following set of approximations were employed exclusively in various ancient civilizations :

(B) $\sqrt{(a^2 + r)} = a + r/(2a+1)$

(C) $\sqrt{(a^2+r)} = a + r/2a$

(D) $\sqrt{\left(a^2+r\right)} = a + r/2a - (r/2a)^2 / \{2(a + r/2a)\}$

(E) $\sqrt{N} = \sqrt{\left(NA^2\right)} / A$, where A is a suitable large number

(F) $\sqrt{N} = \sqrt{\left(NA^2\right)} / A = \sqrt{(P^2 + R)} / A \cong P / A$

(G) $\sqrt{N} = \sqrt{(a^2+r)} = (1/2)\,(a + N/a)$

2.4 NĀRĀYAṆA'S METHOD OF APPROXIMATING QUADRATIC SURDS

Brahmagupta in his Brahmasphuṭasiddhānta (=BS) (see. BS 64, commentary by Pṛthudaka Svāmī (c. 863 A.D.) has considered the following indeterminate quadratic equations :

(H) $Nx^2 \pm c = y^2$

where N is a non-square integer called multiplier; x and y, the smaller and greater roots respectively and c, the interpolator (here the roots are taken to be positive integers). The most fundamental equation used so far by *Bhāskara* II (*Bījagaṇita* V. 71) and *Nārāyaṇa* (c. 1357 A.D.) (*Bījagaṇita* I, 86 (p. 44), see also *Ganita Kaumidī* X, 17 part II (p. 244)) is

(I) $Nx^2 + 1 = y^2$.

The pair of roots of this equation (I) is found by using *Bhāskara* II's Cakravāla or Cyclic Process (*Bījaganita-Vargaprakṛti*, V.2-4). An infinite number of integral (solutions is given by Brahmagupta's first lemma (BSS XVIII, 64). From (I),

$$\sqrt{N} = (y/x)\,\sqrt{1 - 1/y^2}\,.$$

Clearly if y (so also x) is large, y/x is a close approximation to $\sqrt{N}$. A similar method of closer approximation to the value of square-root was given by Euler [16].

2.5 IRRATIONALITY OF $\sqrt{2}$

Bāudhayana, Āpastamba and *Kātyāyana* gave the value of $\sqrt{2} = 1 + 1/3 + 1/(3\text{x}4) - 1/(3\text{x}4\text{x}34)$ (approx.) with an additional term *viśeṣa*.

By giving examples (one of *Sūryaprajñapati*, Sutra 20 and other of *Jambūdvīpa-prajñapati*, Sūtra 3), Bag and Sen (see [5, p. 169]) conclude that the term *viśeṣa* refers to a small quantity which is either in excess or in deficit and cannot be accurately determined. Hence the concept of irrationality of surds (see, for details, [13, pp. 198-202], [50, pp. 44-52] and [59]). Aristotle was the first having given the proof of incommensurability of diagonal of a square with its side [see [8, p. 64], [37, pp, 54-55] and [42, pp. 43-44]).

3. GEOMETRY

[According to S. Kak, An Overview of Ancient Indian Science, Pre-proceedings of Seminar on Science and Technology in Ancient India, April 25-26, 1998, Institute of Oriental Study, Thane: "Seidenberg, by examining the evidence in the *Śatapatha Brāhmaṇa*, showed that Indian geometry predates Greek geometry by centuries. Seidenberg argues that the birth of geometry and mathematics had a ritual origin. For example, the earth was represented by a circular altar and the heavens were represented by a square altar and the ritual consisted of converting the circle into a square of an identical area. There we see the beginings of geometry!"

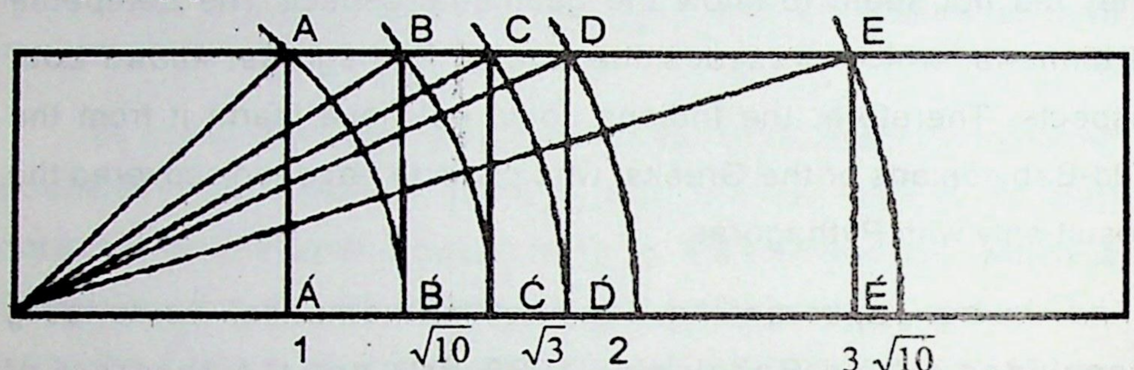
Seidenberg, an authority in ancient mathematics related fields considered two aspects of the "Pythagoras" theorem in the Vedic

literature. One aspect is purely algebraic that presents numbers *a, b, c,* for which $a^2 + b^2 = c^2$. The second is the geometric, according to which the sum of the areas of two square areas of different size is equal to another square. Seidenberg argued that the Babylonians knew the algebraic aspect of this theorem as early as 1700 BC, but they did not seem to know the geometric aspect. The *Śatapatha Bramhaṇa*, which precedes the age of Pythagoras, *knows* both aspects. Therefore, the Indians could not have learnt it from the Old-Babylonians or the Greeks, who claim to have rediscovered the result only with Pythagoras.

In his paper on the origin of mathematics, Seidenberg concluded : "Old-Babylonia (1700 BC) got the theorem of Pythagoras from India or that both Old-Babylonia and India got it from a third source. Now the Sanskrit scholars do not give a date so far back as 1700 B.C. Therefore I (postulate) a pre-Old-Babylonian (i.e., pre-1700 B.C.) source of the kind of geometric rituals we see preserved in the *Śulbasūtras*, or at least for the [mathematics] involved in these rituals." That was before archaeological findings disproved the earlier assumption of a break in Indian civilization in the second millennium BC; it was this assumption of the Sanskritists that led Seidenberg to postulate a third earlier source. Now with our new knowledge, Seidenberg's conclusion of India being the source of the geometric and mathematical knowledge of the ancient world fits in with the new chronology of the texts."

In the *Śulbasūtras*, a surd (root of both square and non-square numbers) is technically called *karṇī or dvikarṇī* $\sqrt{2}\left(=\sqrt{1^2+1^2}\right)$, *tṛkarṇī* $\sqrt{3}\left(=\sqrt{2^2+1^2}\right)$ etc. and *dvitiyākarṇī* $\sqrt{2}/2\left(=1/\sqrt{2}\right)$, *trtayakarṇī*

$\sqrt{3}/3(=1/\sqrt{3})$ etc. In the following figure, $OA=\sqrt{2}$, $OB=\sqrt{3}$, $OC=\sqrt{4}$, $OD=\sqrt{5},\ldots\ldots,OE=\sqrt{10}$ and numbers like $1, \sqrt{2}, \sqrt{3}, 2$, $\sqrt{5}$, ..., 3, $\sqrt{10}$, ..., 4, etc. are represented on the same line. Clearly they had the idea that $\sqrt{2}$ is greater than 1, $\sqrt{3}$ is greater than $\sqrt{2}$ etc. (see [4].



Following are the sets of triplets used in *Śulbasūtras* at various places while constructing rectilinear figures (e.g., rectangle) [4, pp. 9-10] :

a) $n^2+\left(\frac{3}{4}n\right)^2=\left(\frac{5}{4}n\right)^2$ (BSS 1.5; MSS 1.11-1.12)

(i) $n=4,\ 4^2+3^2=5^2$ (BSS 1.13)

(ii) $n=12,\ 12^2+9^2=15^2$ (KSS 2.5)

(iii) $n=20,\ 20^2+15^2=25^2$ (ASS 5.3)

(iv) $n=16,\ 16^2+12^2=20^2$ (ASS 5.3)

b) $n^2+\left(\frac{5}{12}n\right)^2=\left(\frac{13}{12}n\right)^2$ (ASS 1.2; BSS 1.8; KSS 1.4)

(i) $n=1,\ 1^2+\left(\frac{5}{12}\right)^2=\left(\frac{13}{12}\right)^2$ (ASS 1.2; BSS 1.8; KSS 1.4-1.5)

(ii) $n=36,\ 36^2+15^2=39^2$ (BSS 1.13; KSS 5.4)

(iii) $n = 188,\ 188^2 + \left(78\frac{1}{3}\right)^2 = \left(203\frac{2}{3}\right)^2$ (ASS 6.5)

(iv) $n = 6,\ 6^2 + \left(2\frac{1}{2}\right)^2 = \left(6\frac{1}{2}\right)^2$ (ASS 6.6; MSS 2.4)

(v) $n = 5,\ 5^2 + \left(2\frac{1}{12}\right)^2 = \left(5\frac{5}{12}\right)^2$ (ASS 6.7)

(vi) $n = 10,\ 10^2 + \left(4\frac{1}{6}\right)^2 = \left(10\frac{5}{6}\right)^2$ (ASS 6.8)

(vii) $n = 27,\ 27^2 + \left(11\frac{1}{4}\right)^2 = \left(29\frac{1}{4}\right)^2$ (ASS 7.3)

(viii) $n = 18,\ 18^2 + \left(7\frac{1}{2}\right)^2 = \left(19\frac{1}{2}\right)^2$ (ASS 7.1-7.2)

(ix) $n = 12,\ 12^2 + 5^2 = 13^2$ (ASS 5.4; BSS 1.13)

(x) $n = 96,\ 96^2 + 40^2 = 104^2$ (MSS 1.4-1.6)

c) $7^2 + 24^2 = 25^2$ (BSS 1.13)

d) $8^2 + 15^2 = 17^2$ (BSS 1.13; KSS 5.5)

e) $12^2 + 35^2 = 37^2$ (BSS 1.13; KSS 5.5)

f) $1^2 + 3^2 = \left(\sqrt{10}\right)^2$ (KSS 2.4; MSS 3.5)

g) $2^2 + 6^2 = \left(\sqrt{40}\right)^2$ (KSS 2.5)

h) $1^2 + \left(\sqrt{2}\right)^2 = \left(\sqrt{3}\right)^2$ (KSS 2.10)

i) $na^2 = \left[\frac{(n+1)a}{2}\right]^2 - \left[\frac{(n-1)a}{2}\right]^2$ (where a = rational (KSS 6.7)

j) $6^2 + \left(4\frac{1}{2}\right)^2 = \left(7\frac{1}{2}\right)^2$ (MSS 2.5)

k) $1^2 + \left(\sqrt{10}\right)^2 = 11$ (MSS 12.5)

4. BRIEF HISTORICAL DEVELOPMENT OF π

Now it is well-known that, in a circle, the length of circumference (say C) divided by the length of the diameter (said D), i.e., C/D is transcendental. In modern mathematical sciences, this ratio, viz., C/D is denoted by π. William Jones (1700 A.D.) (see [79, p. 312]) was the first having used this Greek symbol for C/D. In Vedic period, C/D appears to have been denoted by *Tritan* (see [21A, pp. 187 & 326]):

Āryabhaṭa I (*Āryabhaṭa* II 10) gives the following rule (see [27]) to obtain surprisingly such a good value of π.

चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणाम्।

अयुतद्वयविष्कम्भस्यासन्नो वृत्तपरिणाहः।।

That is, π = 62832/20000 = 3.1416.

Using the theory of continued fractions, it can be written as (see [27]).

$$\pi = 3 + \frac{1}{7+}\frac{1}{16+}\frac{1}{11}.$$

Successive convergents of this continued fraction give :

(i) 3, the simplest approximation (see, for details, [20], [17, p.9], [51, p.8], [52], [56, p.99], [59], [67, p.857], [69, p.357], [70, p.456], [79, p.302] and [84]).

(ii) 22/7, called Archimedian value (see, for details, [21, p.10], [22, p.172],

[61, p.168], [63], [79, pp.307-310] and [84, p.33]).

(iii) 355/113, called the Chinese value (see for details [8, p.136], [9, p.147], [41, p.42&22], [56, p.101], [68, p.19] and [79, p.310]).

(iv) 3927/1250 = 3.1416 is the reduced form of Ārtbhaṭa I's value. This value is used to calculate the diameter of the earth from its circumference [17] (see, for details, [7, pp.313-314], [9, p.187], [21, p.28] and [61, p.166]).

4.1 A Stanza in *Kriakramkari* (p.377) quotes-the diameter of a circle multiplied by 104348 and divided by 33215 gives the circumference very accurately, i.e., correct to nine decimal places [24]. *Karaṇpaddhatī* (VI, 7) of Putuman Somayjin (1660-1760 A.D.) gives- when the minutes (or parts) of a circumference are multiplied by 10^{10} and divided by 31415926356, we get the diameter, i.e., π correct to 10 decimal places (see *Karaṇpaddhatī* edited by K. Sambasiva *śastri*, Trivendrum, 1937, p.17). *Nīlkantha Somayājin* in his commentary, on *Āryabhaṭīya* quotes *Hādhava's* rule (see *K. Sambasiva Śastrī* [op. cit.]), acording to which

$$\pi = 2827433388233 / (9 \times 10^{11})$$

$$= 3.14159265359 \text{ (correct to 11 decimal places).}$$

4.2 Jaina Value of π

Tiloyasāra (17, p.9), in Prakrit, of Nemicandra (975 A.D.) gives $\pi = 3$ [25]. Its *gāthā* 96 states - the square root of ten times the square of the diameter becomes the circumference of the circle (see [25], that in

$$\text{(J)} \qquad C = \sqrt{(10D^2)}\left(=\sqrt{10}D\right)$$

This implies $\pi = \sqrt{10}$

Hādhava Candra (pupil of *Nemicandra*) in his commentary of *Tiloyasāra*, in Sanskrit, under the above *gāthā* gives the derivation of (J) based on the on consideration of area of a regular octagon [30]. The same is

also obtained by the process of averaging $9 < \pi^2 < 11$ [29]. Here the result $9 < \pi^2 < 11$ is obtained by considering the areas and perimeters of octagon [30] (see, for details, [7], [43] and [56,p. 102]).

4.3 Irrationality of π

After Bhāskara II, *Nīlkantha* realized irrationality *of* π, (*see Āryabhaṭīyabhāsya of Nīlkantha,* under *Gaṇita* (V. 10) and so he gave the value of π in terms of the series. The irrationality of π was first proved by J.H. Lambert in 1766 A.D. In his Elements de Geometrie. He also proved the irrationaluty of π^2 there by giving idea that the value of π can be put in quadratic surd form [32].

4.4 π - Series

Karaṇapaddhati (VI, V.1) given the slowly convergent series

(K) $\theta = \tan\theta - 1/3\tan^3\theta + 1/5\tan^5\theta - \ldots\ldots\ldots, |\theta| \le \pi/4.$

Putting $\theta = \pm\pi/4$, we get

(L) $\pi/4 = 1 - 1/3 + 1/5 - \ldots\ldots\ldots\ldots$

The series (L) is known as *Mādhava -Gregory series* (Gregory,1638-1675 A.D.). Its proof is given in *Yuktibhāṣa* (ch. VI)of Mā dhava(14th Century AD). It is evident that ancient authors were aware of other slowly convergent series. Srinivasa Ramanujan discovered several rapidly convergent series for π [6,pp.191-912]

NOTE

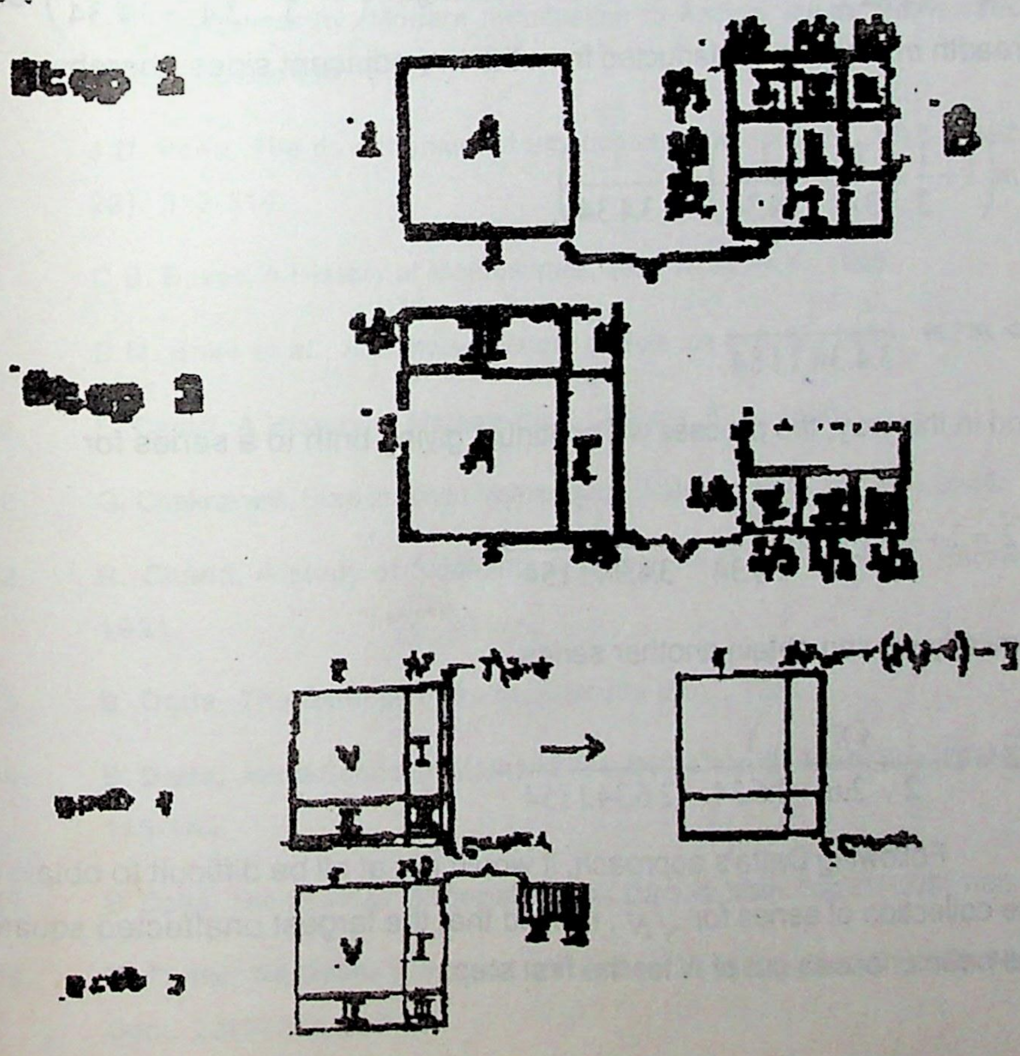
1. Personal communication from S. Das Gupta (Dec. '96): Reader's Digest : Book of Facts Categorily states 'The Theorem - Pythgoras did not invent'. However the right angle triangle has the well know property. This can be stated geometrically by drawing squares, arithmetically by triad as 3,4,5 and algebraically by $x^2 + y^2 = z^2$. Question is who evolved which ? S.R. Rao in his book on Harappa civilization gives description of square sacrificial altar, as such the Indian story goes back to the civilization.

2. Referee's suggestion on a paper computation of $N^{1/n}$ communicated to *Gaṇita Bhārtī* :

The term, '*Karaṇī*' used in the *Śulbasūtras* implies the existence of the method of square-root extraction in the Vedic period. Although the rule for the extraction of square-root is at first found in the *Āryabhaṭīya,* it does not at all mean that *Āryabhaṭa* I is the inventor of the rule, which is evident in the reference to *Maskarī, Putana* etc. who had written books on mathematics in details.

3. Datta's approach for $\sqrt{2}$ and series representation for $\sqrt{N}$:

The theme behind seems to be based on the combination of two unit squares into one, i.e. net area 2 will yield side-length $\sqrt{2}$. In it, one square will be kept as such and various blocks of the other will be cemented to the adjacent sides of the former, the description is self-evident in the first three steps.



To compensate the corner two strips (each of the length $\left(1+\frac{1}{3}+\frac{1}{3.4}\right)$ and breadth 1) have to be deducted from the two adjacent sides, that is,

$$21\left(1+\frac{1}{3}+\frac{1}{3.4}\right)=\left(\frac{1}{3.4}\right)^2$$

$$\Rightarrow 1=\frac{1}{3.4.34}$$

Again to compensate the corner of the overlapping area $\left(\frac{1}{3.4}-\frac{1}{3.4.34}\right)$, two strips (each of the length $\left(1+\frac{1}{3}+\frac{1}{3.4}-\frac{1}{3.4.34}\right)$ and breadth m) have to be deducted from the two adjacent sides, thereby

$$2m\left(1+\frac{1}{3}+\frac{1}{3.4}-\frac{1}{3.4.34}\right)\doteq\left(\frac{1}{3.4.34}\right)^2$$

$$\Rightarrow m = \frac{1}{3.4.34.1154}.$$

And in this way, the process will continue, giving brith to a series for

$$\sqrt{2}=1+\frac{1}{3}+\frac{1}{3.4}-\frac{1}{3.4.34}-\frac{1}{3.4.34.1154}+\ldots.$$

Similarly we can obtain another series

$$\sqrt{2}=1+\frac{1}{2}+\frac{1}{2.6}-\frac{1}{2.6.34}-\frac{1}{2.6.34.1154}-\ldots.$$

Following Datta's approach, it would not at all be difficult to obtain a fine collection of series for $\sqrt{N}$, remind that the largest unaffected square has been choosen out of N for the first step.

REFERENCES

1. V.S. Agarwal (ed.), Vedic Mathematics by *Svāmi Bhārtī Kṛṣna Tirthaji Mahārāja,* Motilal Banarasidass, Delhi, 1965.

2. D. Arkasomayaji (ed.) *Siddhānta Śiromaṇi* of *Bhāskārācrya*. The Rothnam Press, Madras-1, 1980.

3. A.K. Bag. Mathematics in Ancient and Medieval India, Chaukamba Orientalia, Varanasi, 1779.

4. A.K. Bag, Ritual geometry in India and its parallelism in other cultural areas, Indian J. Hist. Sci. 25 (1990), 1-19.

5. A.K. Bag and S.N. Sen (ed.), The *Śulbasūtras*, INSA, New Delhi, 1983.

6. T.S. Bhanumurthy, Modern Introduction to Ancient Indian Mathematics, Wiley Eastern Ltd., 1991.

7. J.D. Bond, The development of trigonometric methods etc., ISIS, 4(1921-22). 313-314.

8. C.B. Boyer, A History of Mathematics, John Wiley, N.Y., 1968.

9. D.M. Bose et al., A Concise History of Science in India, INSA, 1971.

10. F. Cajori, A History of Mathematics, Chelsea, N.Y., 1980.

11. G. Chakravarti, Surd in Hindu Mathematics, J. Dept. Letters 24(1934),29-58.

12. R. Chand, A study of *Siddhānta Śiromāṇi*, Ph.D. Thesis, GKV, Hardwar, 1991.

13. B. Datta, The Science of *Śulba*, Calcutta Univ., 1932.

14. B. Datta, Jaina School of Mathematics, Bull. Calcutta Math. Soc. 21(1929) 115-145.

15. B. Datta, The *Bakhśali* Mathematics, Bull. Calcutta Math. Soc.21(1929).1-60.

16. B. Datta, Nārāyāṇa's method for values of surds, Bull. Calcutta Math. Soc., 23(1931), 187-194.

17. B. Datta, Hindu values of π, J. Asiatic Society Bengal, 22(1928), 25-42.

18. B. Datta and A.N. Singh, History of Hindu Mathematics (Part I and II), Asia Publishing House, Bombay, 1962.

19. L.E. Dickson, History of Theory of Numbers (3 vols), Chelsea, 1971.

20. Davarikadasa (ed.) Vasubhadu's Abhidārmakosam, Buddha Bhāratī, Varanasi, 1981.

21A. K.D. Dvivedi, The Essence of Vedas,Vishva Bhāratī Research Inst., Gyanpur, 1990.

21. S. Dvivedi (ed.)., The Śiṣyadhivṛdhida, Grahagaṇita, Benares, 1886.

22. S. Dvivedi (ed.)., Mahāsiddānta, Benares, 1910.

23. L.V. Gurjar, The value of $\sqrt{2}$ given in the *Śulbasūtras*, J. Univ. Bombay 5 (1942), 6-10.

23A. M.L. Gupta, Astro-Physics in Vedas, Dharam Hinduja International Centre of Indic Research, Delhi, 1984.

24. R.C. Gupta, Baudhāyana's value of $\sqrt{2}$, Math. Edu., 6(1972), 77-79.

25. R.C. Gupta, Circumference of Jambūdvīpa in Jaina Cosmography, IJHS, 10 (1975), 38-46.

26. R.C. Gupta, Some Ancient Values of *Pi* and their use in India, Math. Edu., 9(1975), 1-5.

27. R.C. Gupta, *Āryabhaṭa* I's value of π, Math. Edu. 7(1973), 17-20.

28. R.C. Gupta, New Indian values of π from *Mānava Śulbasūtras*, Centaurus, 31 (1988), 120-123.

29. R.C. Gupta, The process of averaging in ancient and medieval Mathematics, Gaṇita-Bhartī, 3 (1981), 32-41.

30. R.C. Gupta, Mādhava Candra's and other octagonal derivative on Jaina value of $\pi = \sqrt{10}$, IJHS, 21 (1986), 131-139.

31. R.C. Gupta, On the derivation of Bhaskara I's formula for the sine, Gaṇita-Bhartī, 8 (1986), 39-41.

32. R.C. Gupta, Lindemenn's discovery of the transcendence of π : A centenary tribute, Gaṇita-Bhartī, 4(1982), 102-108.

33. R.C. Gupta, Vedic Mathematics form the *Śulbasūtras*, India J. Math. Edu., 9(1981), 1-9.

34. T.B. Hardikar, Indeterminate Analysis (Ancient and Modern, 759/16, Decan Gymkhana Colony, Pune, 1991.

35. T. Hayashi, A new, Indian rule for the squaring of a circle: *Mānava Śulbasūtra* (3.2, 9-10), Gaṇita-Bhartī, 12 (1990), 75-82.

36. T. Hayashi et al., Indian value of π derived from *Āryabhaṭa's* value, Historia Scientiarum 37(1989), 1-16.

37. T.L. Heath, A History of Greek Mathematics, Part I (Oxford, 1921) and part II (Dover, 1981).

38. T.L. Heath, The Thirteen Books of Euclid's Elements (3 vols), Dover, 1953.

39. L.L. Jha (ed.), Līlāvatī of Bhāskarācārya II, The Chowkhamba, Vidya Bhawan, Varanasi, 1961.

40. K.D. Joshi (ed.) *Siddhānta Śiromāni* of Bhāskarācārya, Kashi Hindu Univ. Press, Allabalad, 1961.

41. H.L. Jain (ed.) Satkhandagam with Dhavalā Commentary, Vol. 4, Amravati, 1942.

42. S.Kak, The Nature of Physical Reality, Vitasa Institute, Baton Rouge, 1987.

43. H.R. Kapadia (ed.), *Gaṇita-tilaka* by Śripati with the commentary of Siṁhatilaka Sūri, Baroda Oriental Institute, 1937.

44. S.D. Khadikar (ed.) *Kātyāyana Śulbasūtras*, Vedic Samsodhan Mandala, Poona, 1974.

45. R.S. Lal, Extraction of square-root in ancient Hindu Mathematics, Math. Edu., 18(1984), 130-135.

46. R.S. Lal, Extraction of cube-root in Hindu Mathematics, Math. Edu., 19(1985), 153-160.

47. L.Y. Lam, The geometrical basis of the ancient Chinese square-root method, ISIS, 61(1979), p. 343.

48. L.Y. Lam. A Critical Study of Yang Suan Fa, Singapur Univ., Press, 1977.

49. M. Levey and M. Petruik (ed. and transl.), Principles of Hindu Reckoning by Kushyar Ibn Labban, Wisconsin Univ. Press, Madison, 1965.

50. E. Maor, To Infinity and Beyond, Brikhaüser Boston, Inc., 1987.

51. Y. Mikami, The Development of Mathematics in China and Japan, Chelsea, 1961.

52. A Miler, Approximation in Mathematics regarded as exact, Math. Student, 6(1931, p. 137-138.

53. D.Mishra (ed.), Līlāvatī of Bhāskarācārya, II with Vasana, Mithila Inst. of Post Graduate Studies and Research in Sanskrit learning, Darbhanga, 1959.

54. A. De Morgan, The Encyclopedia of Eccentrices, Open Court, La Salle, 1974.

55. S.A. Naimpally. K.S. Patvardhan and S.L. Singh, Līlāvatī of Bhāskarācārya, Motilal Banarsidass (under publication).

56. J. Needham, Science and Civilization in China Vol. 3 : Mathematics and etc., Cambridge Univ. Press, 1959.

57. O. Neugebauer, The Exact Sciences in Antiquity, Harper, N.Y., 1962.

58. R.A. Parkar, Some demotic mathematical Papyri, Centaurus, 14(1969), 136-141.

59. J. Pottege, The Vitruvian value of π , ISIS, 59 (1968), 190-197.

60. L. Rodet, Surdes methods d' approximation, Chez les anciens, Bull. Soc. Math. France, 7(1879), 159-167.

61. E.D. Sachau(transl.), Al-Biruni's India (two parts in one), Vol I, S. Chand & Co., 1964.

62. S.C. Sarabhai, The cube-root method of Māhavīrācārya, Math. Edu. (1986), 14-18.

63. G. Satron, Introduction to the History of Science (3 vols), Huntington, N.Y., 1975.

64. Satyaprakash and R.S. Sharma (ed.) Baudhāyana *Śulbasūtras*, The Research Inst. of Ancient Scientific-Studies, New Delhi, 1968.

65. Satyaprakash and R.S. Sharma (ed.), Āpastamba *Śulbasūtras*, The Research Inst. of Ancient Scientific Studies, New Delhi, 1968.

66. P.C. Sengupta, The Āryabhatīam, J.Dept. Letters, 16(1927), 1-56.

67. R.S. Sharma (ed.), Brahmasphuṭasiddhānta of Brahmagupta, Indian Institute of Astronomical and Sanskrit Research, New Delhi, 1966.

68. K. V. Sharma (ed.), The Golsāra, Vishveshvaranand Inst. of Sanskrit and Indological Studies, Hoshiarpur, 1970.

69. R. Sharma (ed. and transl.) Vāyu Purāṇa, Part I, Bareily, 1967.

70. R. Sharma (ed.) Kūrma Purāṇa, Part I, Bareilly, 1970.

71. K. S. Shukla, Surds in Hindu Mathematics by Datta and Singh, IJHS, 28(1993), 253-264.

72. K. S. Shukla, Approximate Values of Surds in Hindu Mathematics by Datta and Singh, IJHS, 28(1993), 265-275.

73. K. S. Shukla (ed.), Āryabhaṭiya of Āryabhaṭa, INSA, 1976.

74. B. G. Siddharth, Mahayuga : The great cosmic cycle and date of Ṛg Veda, Research Report, B.M. Birla Science Centre, Hydrabad, 1991.

75. B. G. Siddharth, A lost anatolian civilization-Is it Vedic? Research Report, B. M. Birla Science Centre, Hyderabad, 1992.

76. S. L. Singh, Vedic Geometry, J. Natur.Phys.Sci. 5-8 (1994), 75-82.

77. S. L. Singh and R. Chand, An extension of ancient Indian cube-root method, J. Natur. Phys. Sci., 3(1989), 78-79.

78. A. N. Singh, On Indian method of root extraction, Bull. Calcutta Math. Soc., 18(1927), 123-140.

79. D. E. Smith, History of Mathematics (2 vols), Dover 1958.

80. J. Struik, A Source Book in Mathematics (1200-1800), Harvard Univ. Press, 1969.

81. Svami S. P. Sarasvati, Geometry in Ancient India, Govindaram Hasanand, Delhi, 1987.

82. G. Thibaut, On the Śulbasūtras, J. Asiatic Soc. Bengal, 44(1875), 239-241.

83. B. L. Upadhyaya, Prācīna Bhāratīya Gaṇita Vigyan Bharti, Vajir Nagar, New Delhi, 1971.

84. B. L. Vander Waerden, Science Awakening, Wiley Science Edition, 1963.

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DID VEDIC SAVANTS KNOW IRRATIONAL NUMBERS ?

V. MISHRA* and S. L. SINGH**

This paper presents a brief account of square-root and cube-root methods in Indian perspective. The well-known Gurjar's iterative procedure for obtaining suqare-root of 2 is extended to non-square positive integers, and we conclude that the evaluation of suqare-rooting in *Śulba* and Post-*Śulba* period was most probably based on this extended procedure.

This paper consists of the following sections :

- Introduction
- Computation of $\sqrt{N}$
- Conclusion

INTRODUCTION

1.1 The concept of irrational numbers seems a land mark in the development of science and technology. Modern historians attempt to ascribe this concept to sixth century B.C. philosopher and mathematician Pythagoras (see, for instance [28, p.44]) is not compatible with recent researches in the history of ancient mathematical sciences (see, for instance, [6], [21, pp. 95-97) and [23]). The notion of irrational numbers may be found in *Brāhmaṇas* (ca. 2500 B.C.) and *Saṃhitās* (ca.) 3000 B. C.) (cf. [17], see also

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[12], [19], [30], [31], [32], [36], [38], and [41]). Techniques to compute approximate rational values of certain irrational numbers in *Śulbasūtras* (ca. 1000 - 200 B. C.) is enough evidence that irrationality concept is much older than the period of Pythagoras (ca. 540 B.C.) (see, for instance. [32]). It may be mentioned that *Śulbasūtras* are manuals for constructing altars. This means that *Śulba* rules are much older than their compositions. For details concerning surd terminology, e.g., *dvikaraṇī* $\left(\sqrt{2}\right)$ *tṛkaraṇī* $\left(\sqrt{3}\right)$ etc., refer to [6]. However, a proof to the fact that an irrational number is not expressible as a ratio of two integers had to wait Aristotle (d. 332 B. C.) (see, for a proof, [8, pp. 83 - 85], [24. p. 55] and [28, pp. 43 - 44]).

1.2 The following aphorism of *Baudhāyana Śulbasūtra* (=BSS) 2.12 gives an approximate value of $\left(\sqrt{2}\right)$ (see [32]) :

Pramāṇaṃ tṛtīyena vardhayettacca caturthenātmacatutriṃśonena/ saviśeṣaḥ //

This Means (cf. [32]): The measure is to be increased by its third and this (third) again by its own forth less the thirty -forth part (of that fourth); this is (the value of) the diogonal of a square (whose side is the measure).

That is,

(1.21) $\sqrt{2}$ =1+1/3+1/(3.4) - 1/(3.4.34) (appr.) =577/408.

(This gives correct value to five places of decimal).

The same rule is also quoted in *Āpastamba Śulbasūtra* (=ASS) 1.6 and *Kātyāyana Śulbasūtra* (=KSS) 2.9. Less accurate (i.e., *sthūla*) values of $\sqrt{2}$ are also found in *BSS* [32, p. 174].

On the *Śulba* pattern, commentator *Rāma* (15th century A.D.) gives the following improved approximation [25, p. 85] (see also [36]).

(1.22) $\sqrt{2} = 1+1/3+1/(3.4)-1/(3.4.34)-1/(3.4.34.33)+1/(3.4.34.34)$

$= 577/408 - 1/(3.4.34.33.34)$

$= 647393/457776.$

(This gives correct value to seven places of decimal).

In *BSS* (2.11), *ASS* (3.3) and *KSS* (3.12), an approximate value of $\sqrt{3}$ is frequently used implicitly (see [32, p. 163-164]).

1.3 For the sake of completeness, we cite other approximate values of $\sqrt{2}$ and $\sqrt{3}$ available in other ancient cultural areas.

Babylonian value of $\sqrt{2}$ (ca. 1800 - 1600 B.C.) in fractional presentation may be given as below (cf. [18]) :

(1.23) $\sqrt{2} = 1+24/60+51/60^2+10/60^3$

$= 1.41421296 \ldots .$

Evidently, this babylonian value is a little more accurate than *Śulba* value 1.41421568 ... of $\sqrt{2}$. Many approximations to the value of $\sqrt{2}$ poorer than (1.23) are known in Greek sources [21, Part I, p. 155] (see also [5, p. 132] and [39]).

Ptolemy (200 B. C.) of Greek gives the value of $\sqrt{3}$ which may be expressed in fractions as (cf. [22, pp. 23-24]).

(1.24) $\sqrt{3} = 1+43/60+55/60^2+23/60^3$

$= 1.7320509 \ldots .$

Proof of the method for obtaining approximation (1.21) is not yet known exactly. Of course, several ways of obtaining this formula appears to have been suggested much later (cf. Datta [11], Gurjar [20] and Thibaut [40]; see also [2] - [4], [5], [15] and [32]).

1.4 *Āryabhaṭa* I (b. 476 A. D.) gives an algebraic method of extracting square- and cube-roots of positive integers (see, for instance, [5] and [7], *Mahāvīra* (ca. 850 A.D.), *Śrīdhara* (fl. 850 - 950 A.D.), *Āryabhaṭa* II (ca. 950 A. D.), *Bhāskara* II (b. 1114 A.D.) and *Kamalākara* (fl. 1616 - 1700 A.D.) have also given algebraic methods for extracting suare-roots (see [5, p. 79]). Brahmagupta (b. 598 A.D.), *Mahāvīra Śrīdhara*, *Āryabhaṭa* II and *Bhāskara* II and *Kamalākara* have attempted to give similar algebraic methods for obtaining cube-roots (see [5, p. 80]. For details on extraction of roots, refer to [1], [9], [10], [12], [13], [14], [26], [27], [29], [33], [34] and [35]. Square- and cube-root methods available in Hindu and Jaina works are also found in Arabian works from 9th century A.D. and onwards [37, Part I, pp. 138 - 139]. In the sixteenth century A.D. both square- and cube-roots were given by Canteneo which are exactly the same as those of *Āryabhaṭa* I [37, Part II, p. 148]. Moreover, modern methods of square- and cube-root extraction are simply reduced form of *Āryabhaṭa* I's methods, It seems that essentially Indian methods of root extraction travelled to Europe via Arab and returned back to India from Europe in a slightly cosmetized form. In traditional learning schools of ancient Indian astronomy, these methods of square-and cube-rooting (especially from *Līlāvatī* of *Bhāskara* II) are still taught and used in India.

It would not be amiss to mention that square- and cube-root methods as available basically in ancient Indian mathematical compositions are essentially based on the inversion theme of the formulae $(x+y)^2 = x^2 + 2xy + y^2$ and $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ (see [10] and [35]).

2. Computation of $\sqrt{N}$

2.1 With a view to improving accuracy of approximative calculations

obtained by algebraic methods, ancient Indian mathematicians appear to use iterative procedures while solving equations etc. Brahmaghpta in his *Khaṇḍakhādyaka* (c. 655 A. D.) gives an iterative rule for finding the arc or angle, when its *sine* is known (see [16]. This method is frequently used in Indian system of astronomical calculations (see, for example, Mahābhāskarīya, cf. [16]).

2.2 Proposition. Let $N > 0$ be a non-square integer such that $N = A_1^2 + r$, where A_1 is the largest positive integer and $|r|$ the smallest number. Then

(2.21) $$A_1 = \frac{Nd_{i-1}^2 + n_{i-1}^2}{2d_{i-1}n_{i-1}}, i = 1,2,3, \ldots$$

are the successive convergents to the series

(2.22) $$\sqrt{N} = A_1 + \frac{r}{m} + \sum_{i=3}^{\infty} \frac{Nd_{i-1}^2 - n_{i-1}^2}{2d_{i-1}n_{i-1}}$$

where d_{i-1} *and* n_{i-1} stand for the sum of denominator and numerator upto (i-1) terms respectively and m for a positive integer.

Proof. Let

(2.23) $$\sqrt{N} = A_1 + R_1$$

where R_1 stands for the remainder term.

Note that $N = (A_1 + R_1)^2$. On expanding right hand side,

$$N = A_1^2 + 2A_1R_1 + R_1^2$$

This gives $$R_1 = \frac{N - A_1^2}{2A_1} - \frac{R_1^2}{2A_1} = \frac{r}{2A_1} - \frac{R_1^2}{2A_1}.$$

Suppose $\dfrac{r}{2A_1} = \dfrac{r}{m} + A$.

Then

$$R_1 = \frac{r}{m} + R_2 \; and \; R_2 = A - \frac{R_1^2}{2A_1}$$

Substituting this value of $\boldsymbol{R_1}$ in (2.23) yields

(2.24) $\sqrt{N} = A_1 + \dfrac{r}{m} + R_2$.

That is

(2.25) $\sqrt{N} = A_2 + R_2$ where $A_2 = A_1 + \dfrac{r}{m}$.

Now squaring (2.25), we obtain

$$R_2 = \frac{N - A_2^2}{2A_2} + R_3 \text{ where } R_3 = -\frac{R_2}{2A_2}$$

and, therefore from (2.25),

(2.26) $\sqrt{N} = A_1 + \dfrac{r}{m} + \dfrac{N - A_2^2}{2A_2} + R_3$.

That is

(2.27) $\sqrt{N} = A_3 + R_3$

where $A_3 = A_1 + \dfrac{r}{m} + \dfrac{N - A_2^2}{2A_2}$.

Repeating the above iterations process gives

(2.28) $$\sqrt{N} = A_1 + \frac{r}{m} + \sum_{i=3}^{\infty} \frac{N - A_{i-1}^2}{2A_{i-1}}$$

and the *ith* convergent is given by

(2.29) $$A_i = \frac{N + A_{i-1}^2}{2A_{i-1}}$$

Taking A_{i-1} equal to n_{i-1}/d_{i-1} in (2.28) and (2.29) establishes the Proposition.

EXAMPLE.

For $N = 2$; $A_1 = 1$; $r = 1$, choose $m = 3$. Now we shall apply our Proposition here.

$$A_2 = A_1 + \frac{r}{m} = 1 + \frac{1}{3} = \frac{4}{3}.$$

(Notice that $d_2 = 3, n_2 = 4$

$$A_3 = A_1 + \frac{r}{m} + \frac{Nd_2^2 - n_2^2}{2d_2n_2} = 1 + \frac{1}{3} + \frac{2}{2.3.4} = \frac{17}{12}; (d_3 = 12, n_3 = 17).$$

$$A_4 = A_1 + \frac{r}{m} + \frac{Nd_2^2 - n_2^2}{2d_2n_2} + \frac{Nd_3^2 - n_3^2}{2d_3n_3}$$

$$= 1 + \frac{1}{3} + \frac{2}{2.3.4} - \frac{1}{2.12.17} = \frac{577}{408};\ (d_4 = 408, n_4 = 577).$$

$$A_5 = A_1 + \frac{r}{m} + \frac{Nd_2^2 - n_2^2}{2d_2n_2} + \frac{Nd_3^2 - n_3^2}{2d_3n_3} + \frac{Nd_4^2 - n_4^2}{2d_4n_4}$$

$$= 1 + \frac{1}{3} + \frac{2}{2.3.4} - \frac{1}{2.12.17} - \frac{1}{2.408.577} = \frac{665857}{470832}$$

Therefore we obtain a remarkable infinite series representation

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} - \frac{1}{3.4.34.1154} + \ldots \qquad (2.30)$$

The successive convergents are :

$$A_1 = 1,\ A_2 = \frac{4}{3} = 1.3333333, \qquad A_3 = \frac{17}{12} = 1.4166667,$$

$$A_4 = \frac{577}{408} = 1.4142157, \qquad A_5 = \frac{665857}{470832} = 1.4142316, \text{ etc.}$$

If we take $m = 2$ in the above example, the corresponding series becomes

$$\sqrt{2} = 1 + \frac{1}{2} - \frac{1}{2.6} - \frac{1}{2.6.34} - \frac{1}{2.6.34.1154} + \ldots. \qquad (2.31)$$

Thus we observe that a large number of infinite series representations for $\sqrt{2}$ are available for different values of m.

An analogous treatment will yield a large variety of infinite series representations for $\sqrt{3}$. For example for $m = 3$,

$$\sqrt{3} = 1 + \frac{2}{3} + \frac{1}{3.5} + \ldots.. \qquad (2.32)$$

3. CONCLUSION

The evaluation of square-root of non-square integers in Śulba period seems to be very much based on the series (2.22) with $m = 2A_1 + 1$ (odd). The reasons being :

(a) The series (1.21) with $m = 2.1 + 1 = 3$ takes exactly the same form as (2.30).

(b) *ASS* (3.3), *BSS* (2.11) and *KSS* (3.12) give a rule for quadrature of a circle that mathematically takes the form :

$$2a = d - \frac{2}{15}d\left(=\frac{13}{15}d\right) \tag{3.1}$$

wherein $2a$ is the side of the square and d its diameter.

According to [11. pp. 146-47] (see also [32, pp. 162 - 164]), the rule (3.1) corresponds to

$$2a = \frac{\sqrt{3}}{2}d$$

if we take $\sqrt{3} \approx 1+\frac{2}{3}+\frac{1}{15}=\frac{26}{15}$.

This pattern of $\sqrt{3}$ is akin to (2.32). (Notice that here $m = 2\times 1+1=3$.

The geometers of the post-*Śulba* period (including Jainas and *Bakhśālī* Manuscript) use the same series (2.22) with $m = 2A_1$ (even). For example to evaluate $\sqrt{10}$ "Jainas" prefer $m = 2\times 3 = 6$.

Why ancients have not used $m = 2A_1 + k$ ($k \neq 0,1; k$ being an integer) is a matter of great concern. This untold computational tradition needs further investigation.

REFERENCES

1. V.S. Agarwal (ed.), *Vedic Mathematics by Svāmī Bhārtī Kṛṣna Tīrthajī Maharaja*, Motilal Banarsidass, Varanasi, 1965.
2. A. Ahmad, *The Vedic Principle for approximating square-root of two*, *Gaṇita Bharati* 2(1980), 16-19
3. A. Ahmad, *On Vedic square-circle conversion*, *Gaṇita Bhārtī* 4(1982), 72-77.

4. A Ahmad, *On Babylonian and Vedic square-root of 2, Gaṇita Bhāṛāti* 16 (1994), 1-4.

5. A.K. Bag, *Mathematics in Ancient and Medieval India*, Chaukhambha Orientalia, Varanaṣi, 1979.

6. A.K. Bag, *Ritual geometry in India and its parallelism in other cultural areas*, Indian J. Hist. Sci. 25(1990), 4-19.

7. T. S. Bhanumurthy, *A Modern Introduction to Ancient Indian Mathematics*, Wiley Eastern Limited, New Delhi, 1992.

8. C.B. Boyer, *A History of Mathematics* (revised by Uta C. Merzbach), John Wiley & Sons, New York, 1989.

9. G. Chakravarti, *Surd in Hindu mathematics*, J. Dept. Letters 24 (1934), 59-76.

10. R. Chand, *A Study of Siddhānta Śiromaṇi, Ph.D. Thesis*, Gurukula Kangri University, Hardwar, 1991.

11. B. Datta, *The Science of the Śulba*, Calcutta University, 1932 (reprinted : 1991).

12. B. Datta and A.N. Singh, *History of Hindu Mathematics*, Two Parts, Asia Publishing House, Bombay, 1962.

13. B. Datta and A.N. Singh, *Surds in Hindu mathematics* (revised by K.S. Shukla), Indian J. Hist. Sci. 28(1993), 253-264.

14. B. Datta and A. N. Singh, Approximate *values of surds in Hindu mathematics* (revised by K. S. Shukla), Indian J. Hist. Sci. 28(1993), 265-275.

15. R. C. Gupta, *Baudhāyana's* value of $\sqrt{2}$, Math. Edu. (Siwan) 6(1972), Sec. B, 77-79.

16. R. C. Gupta, Some important Indian mathematical methods as conceived in Sanskrit language, Indological Studies 3(1974), 49-62.

17. R.C. Gupta, Mathematics of the *Mahāvedī*, Vishveshvaranand Indological Journal 22(1984), 1-9.

18. R.C. Gupta, *Indian mathematical sciences abroad during pre-modern times*, Indian J. Hist. Sci. 22(1987), 240-246.

19. R.C. Gupta, *Vedic mathematics from the Śulbasūtras*, Indian Journal of Mathematics Education 9(1989), 1-10.

20. L. V. Gurjar, *Ancient Indian Mathematics and the Vedha*, Continental Book Service, Poona, 1947.

21. T. L. Health, *A History of Greek Mathematics*, Two volumes, Oxford, 1921.

22. T. L. Heath, *A Manual of Greek Mathematics*, Dover, New York, 1963.

23. S. Kak, *The roots of science in India*, India International Centre Quarterly 13(1986), 181-196.

24. S. Kak, *The Nature of Physical Relaity*, Vitasta Institute, Baton Rouge, 1987.

25. J. N. Kapur, *Fascinating World of Mathematical Sciences*, Volume VII (Biography and History of Mathematics), Mathematical Sciences Trust Society, New Delhi, 1990

26. R. S. Lal, *Extraction of square-root in ancient Hindu mathematics*, Math. Edu. (Siwan) 18 (1984), 130-135.

27. R. S. Lal, *Extraction of cube-root in Hindu Mathematics*, Math. Edu. (Siwan) 19(1985), 153-160.

28. E. Maor, *To Infinity and Beyond*, Birkhaüser, Boston, 1987.

29. S. C. Sarabhai, *The cube-root method of Mahāvīrācārya*, Math. Edu. (Siwan) 20 (1986), 14-18

30. A. Seidenberg, *The origin of mathematics*, Archieve for History of Exact Sciences 18(1978), 301-342.

31. A. Seidenberg, *The Geometry of the Vedic Rituals in : AGNI, Volume II* (*Ed. Frist Staal*), Asian Humanities Press, Berkeley, 1983, pp. 95-126.

32. S. N. Sen and A. K. Bag (ed. and transl.), *The Śulbasūtras*, Indian National Science Academy, New Delhi, 1983.

33. K. S. Shukla and K.V. Sarma (ed. and transl.), *Āryabhaṭīya of Āryabhaṭa*, Indian National Science Academy, New Delhi, 1976.

34. A. N. Singh, On Indian method of root extraction, Bull. Cal. Math. Soc. 18(1927), 123-140.

35. S. L. Singh and R. Chand, *An extension of ancient Indian cube-root method*, J. Natur. Phys. Sci. 3(1989), 78-89.

36. S. L. Singh, *Vedic geometry*, J. Natur. Phys. Sci. 5-8(1991-1994), 75-82.

37. D. E. Smith, *History of Mathematics*, Two Volumes, Dover, New York, 1958.

38. C. N. Srinivasiengar, *The History of Ancient Indian Mathematics*, The World Press Limited, Calcutta, 1967.

39. J. Stillwell, *Mathematics and Its History*, Springer-Verlag, New York, 1989.

40. G. Thibaut, *On the Śulbasūtras*, Journal of Asiatic Society of Bengal 44(1875), 227-275.

41. B. L. Upadhyaya, *प्राचीन भारतीय गणित*, विज्ञान भारती, नई दिल्ली, 1971.

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट, खण्ड १ (१९९८), अंक २, २०५-२१२
Gurukula Kangri Vijñāna Patrikā Āryabhaṭa, Vol. 1 (1998), NO. 2, 205-212

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METHOD OF DIVISION OF SECOND DEGREE POLYNOMIALS IN TWO OR THREE VARIABLES BY FIRST DEGREE POLYNOMIAL IN TWO OR THREE VARIABLES

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Swami Bharati Krishana Tirthaji has explained methods of division of polynomials by *Paravartya Sutra*. This paper further extends Swamiji's method of division of second degree polynomials in two or three variables.

KEY WORDS:

Horner, Swamiji, *Paravartya yojayet Sutra.*

INTRODUCTION

The Horner's method of synthetic division for division of polynomials in one variable (say x) is well known with the fact that this method is restricted to linear divisors of the type $ax+b$, a and b are constants.

In the book named 'VEDIC MATHEMATICS' author Swami Bharati Krishana Tirthaji has explained methods of division of polynomials by *Paravartya Yojayet Sutra* where the restriction on degree of divisors is removed and divisor may be any lesser degree polynomial than dividend.

This paper further extends Swamiji's method, to division of second degree polynomials in two or three variables where dividend is second degree

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polynomials in two or three variables and divisor is first degree polynomial in one variable.

I intend to discuss the method in four cases after explaining the Swamiji's method.

SWAMIJI'S METHOD

The Swamiji's method for division of *n*th degree polynomial by *m*th degree polynomial $(m < n)$ can be summarized as follows.

Dividend : $b_0x^n+b_1x^{n-1}+b_2x^{n-2}+ \ldots\ldots\ldots\ldots\ldots +b_n.$

Divisor : $a_0x^m+a_1x^{m-1}+a_2x^{m-2}+ \ldots\ldots\ldots\ldots\ldots +a_m.$

The digits of modified divisor (M.D.) by *Paravartya Yojayet Sutra* are

$-a_1/a_0, -a_2/a_0, -a_3/a_0, \ldots\ldots\ldots\ldots\ldots, -a_m/a_0.$

The tabular form of process of division is :

M.D.	x^n	x^{n-1}	x^{n-2}			x^0	
$-a_1/a_0,$	b_0	b_1	b_2		b_m	b_n	
$-a_2/a_0,$		c_1	c_2		c_m	b_n	
....							
$-a_m/a_0$			d_1	d_2	d_{m-1}	d_m	
						upto $(n+1)^{th}$ column	
	k_0	k_1	k_2	k_{n-m}	k_m	k_{m+1}	k_n

METHOD OF DIVISION OF SECOND .....

Where $k_0=b_0, c_1=-k_0a_1/a_0, k_1=b_1+c_1, c_2=-k_0a_2/a_0,$ etc.

Thus

Quotient $Q=(1/a_0)\{k_0x^{n-m}+k_1x^{n-m-1}+..............+k_{n-m}\}.$

Remainder $R=\{k_m x^{m-1}+k_{m+1}x^{m-2}..............+k_n\}.$

Now we study division of second degree polynomial in two or three variables by first degree polynomial.

CASE-1

When dividend is second degree homogeneous polynomial in x and y and divisor is first degree polynomial in x and y.

Dividend : $ax^2+hxy+by^2$

Divisor : $mx+ny$

The table of process of division is as follows.

M.D.	x^2	xy	y^2
	a	h	b
$-n/m$		e_1	e_2
	a	a_1	r

Where $e_1=-an/m,\ a_1=h+e_1,\ e_2=-a_1n/m,\ r=b+e_2.$

Quotient $Q=(1/m)\{ax+a_1y\}$

Remainder $R=ry^2.$

CASE-2

When the dividend is second degree non homogeneous polynomial and divisor is first polynomial in x and y.

Dividend : $ax^2+hxy+by^2+gx+fy+c$ and

Divisor : $mx+ny+p$.

The table of process of division is as follows.

M.D.	x^2	xy	x	y^2	y	constant
	a	h	g	b	f	c
$-n/m$		e_1	e_2			
$-p/m$			e_3	e_4		
					e_5	e_6
	a	a_1	a_2	r_1	r_2	r_3

Where $e_1 = -an/m$, $a_1 = h+e_1$, $e_2 = -ap/m$, $e_3 = -a_1 n/m$, etc.

Quotient $Q= (1/m)\ \{ax+a_1y+a_2\}$

Reminder $R= r_1y^2+r_2y+r_3$.

[Note the placement of e_5 and e_6].

CASE-3

When the dividend is second degree homogeneous polynomial and divisor is first degree polynomial in x,y and z.

Dividend : $ax^2+hxy+by^2+gxz+fyz+cz^2$.

Divisor : $mx+ny+pz$.

The table of process of division is as follows.

M.D.	x^2	xy	xz		y^2	yz	z^2
	a	h	g		b	f	c
$-n/m$		e_1	e_2				
$-p/m$			e_3		e_4		
						e_5	e_6
	a	a_1	a_2	r_1	r_2	r_3	

Where $e_1 = -an/m$, $a_1 = h+e_1$, $e_2 = -ap/m$, $e_3 = -a_1n/m$, etc.

$e_4 = -a_1p/m$, $a_2 = g+e_2+e_3$, etc.

Quotient $Q = (1/m)\ \{ax+a_1y+a_2z\}$

Reminder $R = r_1y^2+r_2yz+r_3z^2$.

[Note the placement of e_5 and e_6].

CASE-4

When the dividend is second degree polynomial (non homogeneous) in three variables x,y,z and divisor is first degree polynomial in x,y,z.

Dividend : $ax^2+by^2+cz^2+hxy+gxz+fyz+ux+vy+wz+d$ and

Divisor : $mx+ny+pz+t$.

The table of process of division is as follows.

M.D.	x^2	xy	xz	x	y^2	yz	y	z^2	z	d
	a	h	g	u	b	f	v	c	w	d
$-n/m$		e_1	e_2	e_3						
$-p/m$					e_4	e_5	e_6			
$-t/m$						e_7	~	e_8	e_9	
							e^{10}	~	e^{11}	e^{12}
	a	a_1	a_2	a_3	r_1	r_2	r_3	r_4	r_5	r_6

Where e_1 to e_{12}, a_1 to a_3 and r_1 to r_6 are calculated similar to previous cases

Quotient $Q = (1/m)\ \{ax+a_1y+a_2z+a_3\}$

Reminder $R= r_1y^2+r_2yz+r_3y+r_4z^2+r_5z+r_6.$

ILLUSTRATIONS :

Ex.1 [Swamiji's Method]

Divide $6x^6-12x+7x^5+23x^4-16x^3-17x^2-3$

by $2x^3-4x^2+5x+3.$

M.D.	x^6	x^5	x^4	y^3	x^2	x	constant
	6	-12	7	23	-16	-17	-3
2		12	-15	-9			
-5/2			0	0	0		
-3/2				-16	20	12	
					-4	5	3.
	6	0	-8	-2	0	0	0

Quotient $Q= (1/2)\ \{6x^3+0x^2-8x-2\}$

$= 3x^3-4x-1.$

Reminder $R= 0.$

Ex.2 [Case 3]

Divide $2x^2-13xy+(1/7)y^2-4xz-(63/2)yz+(3/8)z^2$

by $2x+3z.$

Ans. We write $2x+3z = 2x+0y+3z$

M.D.	x^2	xy	xz	y^2	yz	z^2
	2	-13	-4	1/7	-63/2	3/8
0		0	3			
-3/2				0	39/2	
				0		3/2
	2	-13	-1	1/7	-12	15/8

Quotient $Q= (1/2) \{2x-13y-z\}$

Reminder $R= (1/7)y^2 - 12yz+(15/8)z^2$

Ex.3 [Case 4]

Divide $x^2-6y^2+z^2-5xy+2xz+4yz+3x-2y+8z+4$

by $x+y+2z-1$.

M.D.	x^2	xy	xz	x	y^2	yz	y	z^2	z	d
	1	-5	2	3	-6	4	-2	1	8	4
-1		-1	-2	1						
-2					6	12	-6			
1						0	~	0	0	
							-4	~	-8	4
	1	-6	0	4	0	16	-12	1	0	8

Quotient $Q= x-6y+4$

Reminder $R= 16yz-12y+z^2+8$

The above discussion clearly reflects the importance of Swamiji's method and its extensive use in case of division of second degree polynomials in two or three variables. These procedures are also useful in two and three dimensional geometry.

REFERENCES

1. Higher Algebra : Hall and Knight.

2. Vedic Mathematics : Swami Bharti Krisna Tirtha (Motilal Banarasidas)

3. Vertically and Crosswise : A.P. Nicholas and K. Williams.

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट, खण्ड १ (१९९८), अंक २, २१३-२२२
Gurukula Kangri Vijñāna Patrikā Āryabhaṭa, Vol. 1 (1998), NO. 2, 213-222

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REMARKS ON CERTAIN APPROXIMATION TECHNIQUES OF *GAṆITATILAKA*

V. ARORA* and V. GOEL*

The main purpose of this paper is to discuss and compare techniques for solving equations $x^2 = a$ and $x^3 = a$ (a is any positive integer) as provided in *Gaṇitatilaka* and elsewhere. Also we have extended this technique to nonperfect and decimal number.

The *Gaṇitatilaka* is one of the works of *Pāṭī -ganita* of *Vedic* traditions.

Vedic Mathematics is supposed to be earlier than the ancient Mathematics of Babylonia, China and Egypt see [5].

Ṛgveda, *Yajur-veda*, *Sāma-veda*, *Atharva-veda*, *Saṁhitas* (namely, *ṚK, Sāma, Yajus and Atharvan)*, *Brāhmaṇans*, *Āraṇẏakas*, *Upaniṣads*, *Six Vedāngas* (namely, *śikhṣa*, *Kalpa*,. *Vyākaraṇa*, *Nirukta*, *Chanda and Jyautiṣa)*, *Upvedas* (namely, *Āyurveda*, *Dhanur-Veda*, *Gāndharva-veda and Sthāpatya-veda),the Prātiśākhyas*, *Baudhāyanaśulbasūtra*, *Āpastambhaśulbasūtra*, *Kātyayanśulbasūtra*, *Mānavśulbasūtra*, *Maitrāyanaśulbasūtra*, *Varāhaśulbasūtra*, *Hiraṇyakeiśulbasūtra* and Pāninisūtra are the various famous remarkable sources of *Vedic* Mathematics.

Āryabhaṭa I (b.476 A.D.), *Brahmagupta* (b.598 A.D.),

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Bhāskarā I (c.629 A.D.) *Śridhara* (c.750 A.D.) *Mahāvīracarya* (c.850 A.D.) *Āryabhaṭa* II, (c.950 A.D.), *Śrīpati* (1039 A.D.) and *Nārayāṇa Paṇḍit* (c.1356 A.D.) are the famous ancient Indian mathematicians.

The authour of book *Gaṇitatilaka*, *Śrīpati* was most prominet Indian mathematician, born in śaka 921 (999 A.D.) at *Rohinīkhaṇda*. He was son of *Nāgadevbhaṭṭa* and grandson of *Keśavabhaṭṭa*. *Śrīpati* was *Maheśvara* or *Śaiva* by Religion and a *Kāśyapa* by lineage. He composed his mathematical and astronomical treatises in *Rohinīkhaṇda* as it is clear from the following verses :

'भट्टकेशवपुत्रस्य नागदेवस्य नन्दनः ।
श्रीपती रोहिणीखण्डे ज्योतिः शास्त्रमिदं व्यधात् ।।'
"कश्यप' वंशपुण्डरीकखण्डमार्तण्डः केशवस्य
पौत्रः नागदेवस्य सूनुः श्रीपतिः संहितार्थमाभिधातुमिच्छुराह"

In *Gaṇitatilaka Srīpati* first offers his salutations to God as it is clear from the following verse:

रूपोज्झितं रूपयंतं, स्वरूप-मात्मस्वरूपं परमं प्रणम्य ।
करोमि लोकव्यवहारेहेतो-र्विचित्रवृत्तं गाणितस्य पाटीम् ।।

Which means [12,p. 114]

"Heaving bowed to the supreme who is forsaken as well as possessed of from, who is his own nature, (and also) the nature of the soul itself. I prepare this Arithmetic, having variegated verses for public use".

Gaṇitatilaka consist of certain computational techniques such as computation of cube, cube root etc. In this paper this techniques is extend for non-perfect and decimal number.

This book contains different methods of operations on fractions which have been compared with the modern techniques of operations of fractions.

Gaṇitatilaka gives some salient features of pre-basic and basic comcepts of coinage.

It also gives historical back ground of weight measurement which was used to exist in ancient India, chronologically.

Ancient Indian mathematicians, *Āryabhaṭa* I (b.476 A.D.) [14] *Brahmagupta* (b.598 A.D.) [4], *Bhāskarā* I (c.629 A.D.) [10], *Śrī dhara* (c.750 A.D.) [3], *Mahāvī ra* (c.850 A.D.) [11] *Nārāyaṇa Paṇḍit* (c.1356 A.D.) [5], and also other Indian mathematicians [1,2] give the identical methods to find out the cube-root of a perfect number (in this paper, we define a number as a perfect number of order n if it is expressible in the form of x^n, n being a positive integer and x a rational number). An attempt has also been made by Lal [9] to give a chronological description of the methods given by the Hindu mathematicians for obtaining the real cube-rood of an integer.

Śrī pati in his *Gaṇitatilaka* gives a method to find out cube root of a positive integer and cube root of a fraction only. He has not discussed the methods to find the cube root of number which are not perfect and that of decimal numbers.

Now we discussed some computational techniques from *Gaṇitatilaka:*

(a) COMPUTATIONAL TECHNIQE OF $x^3=a$ (WHERE a IS ANY POSITIVE INTEGER)

Śrī pati method of cubing is given by following slokas :

स्थाप्यो घनोऽन्त्यस्य कृतिश्चं तस्य।
त्रिकादिनिघ्नी कृतिरादिमस्य।
अन्त्यत्रिनिघ्नादिघनश्च सर्वे
स्थानाधिकत्वं मिलिता धनः स्यात्।।

Which means (see [12, p. 117] :

The cube of the last digit, its square multiplied by three and the preceding digit, the square of the preceding digit multiplied by the last digit and by three, and the cube of the preceding digit. All these are united one place in excess of one other. The cube should result.

This method is illustrated by the following example taking digits of different nature.

Example : Compute the cube of 317.

Here last digit is 7, let it be y. Then preceding digit is 31, let it be x, following *Śrīpati,* the cube of the last digit, i.e., $y^3=7^3=343$. The square of last digit multiplied by three & the preceding digit, i.e., $3xy^3=3.31.7^2=4557$. The square of the preceding digit multiplied by the last digit & by three, i.e., $3x^2y=3.31^2.7=20181$. Cube of the last digit, i.e., $x^3=31^3=29791$, which can again be calculated by the same technique using $y = 1$ & $x = 3$.

Adding the above results as 343 units, 4557 tens, 20181 hundreds and 29791 thousands, we can obtain the cube of 317 as 31855013.

(b) COMPUTATIONAL TECHNIQUES FOR $x=\sqrt[3]{a}$ (WHERE x IS ANY POSITIVE INTEGER)

The author *Śrīpati* of *Ganitatilaka* has given the following method for extracting cube root, which is given by the following slokas :

घनोऽघनद्वन्द्वमिति प्रपात्य घनं घनानन्मूलमघः पदस्य।
नयेत् तृतीयस्य हरेच्च शेषं त्रिनिघ्नकृत्यास्य नियोज्य लब्धम्।।
पङ्क्तयां ततस्तत्कृतिमन्त्यनिघ्नीं त्रिसङ्गुणां चापनयेद् घनं च।
विधानमेतद् गणकेन नूनं पुनर्विधेयं घनमूललब्ध्यै।।

Which means (see [12, p. 118])

The places starting from the units place are to be divided.

Thus : One cubic place and then a pair of non-cubic places. Having subtracted the greatest possible cube from the last ghana place, one should bring the cube-root under the third place of the resulting number and divide out the remainder, i.e., the number to the left of the cube-root by thrice its square. Having placed the quotient in the line of the root, one should then subtract the square of this as multiplied by thrice the last numbers, i.e., the number to the left of the quotient in the line of the root, from the number above. This operation is surely to be performed by the ganak repeatedly, for the obtainment of the cube-root.

This method is illustrated by the following example.

Example: Compute the cube root of 1981385216.

First we divide the give number into cubic and non cubic places. The digit at the unit place is a cubic digit and the next two digits are non cubic digits. The rest of the digits in the number can be arranged in the similar way as $\begin{matrix}\sigma & \delta & \delta & \sigma & \delta & \delta & \sigma & \delta & \delta & \sigma \\ 1 & 9 & 8 & 1 & 3 & 8 & 5 & 2 & 1 & 6\end{matrix}$

Here the sigma (σ) represents cubic places and delta (δ) non cubic places.

It can be seen that the digit at tenth place is 1 which is the last cubic digit. Subtracting the greatest possible cube less than or equal to 1, i.e., 1($=a^3$), (i.e. a=1) from 1 we get 0. Now we take the remaining digit of the number one by one from left to right. Thus we take the next digit, namely, 9. Next, we divide 9 by $3a^3$, i.e., 3. The quotient obtained by this division is b (=2) and the remainder is 3. We take the next digit from the number and make the 3 as 38. From this 38, we substract $3ab^2$, i.e., 12 and obtain 26. We again take the next digit from the number and make the 26 as 261. We again take the next digit from the number and make the 26 as 261. Now, We subtract b^3(=8) from 261 and obtain 253. We take the next digit from the

number, i.e., 3 and divide 2533 by 3 $(10a+b) = 432$. The quotient, thus obtained is c which is equal to 5 here and the remainder comes out to be 373. Taking the next digit from the number, we have 3738. Now $3(10a+b)c^2$, i.e., 900 is subtracted from this number which gives us 2838. Again, taking the next digit from the number 2838 comes 28385. Again, taking the next digit from the number 2838 comes 28385, we subtract $c^3 (=125)$, from this number the remainder becomes 28260. In the next operation, we take the next digit from the number, i.e., 2 and divide 282602 by $3(100a+10b+c)^2=46875$. The quotient thus obtained is d which is equal to 6 and the remainder comes out to be 1352. Carrying the next digit of the number we have 13521. Now $3(100a + 10b+c)d^2$, i.e., 13500 is subtracted from this number which gives us 21. Again, taking the next digit 21 becomes 216. Next, we subtracted d^3 (=216) from this number and thus the difference is zero and the procedure terminates. As such, we have the cube root of 1981385216 from this process as $1000a+100b+10c+d = 1256$.

The following observations can be noted in this method.

(i) The total number of Sigmas in the given number gives the total number of digits in its cube-root.

(ii) For a number having two Sigmas, the cube-root is 10a+b, where b is the unit place of the number.)

(iii) For three Sigmas in a number, the cube root is 100a+10b+c, (where c is the unit place of the number).

(iv) For four Sigmas in a number, the cube root is 1000a+100b+10c+d, (where d is the unit place of the number).

The technique of finding the cube root can also be applied to the numbers which are not having the exact cube root. The

following example illustrate this fact.

Example : Compute the cube root of 3.

The given number is not perfect in powers, so after unit place let us put decimal and zeroes in the end of the remainders for further calculations as shown below. Put sigma and deltas as mentioned in the above method.

Here the number upto cubic place is 3. We observe that the greatest possible cube less than or equal to 3 is 1. So we claim that $a^3=1$, i.e., $a=1$. Subtracting 1 from 3 we get 2. Now we take zero's as remaining digits. Thus putting 0 after 2, we make it 20. Next, we divide 20 by $3a^2$, i.e., 3. The quotient obtained by this division is b (=4) and the remainder is 8. We take 0 as next digit and make the 8 as 80. From the 80, we subtract $3ab^2$, i.e., 48 and obtain 32. Again we take 0 as next digit and make 32 as 320. We subtract $b^3(=64)$ from 320 and obtained 256. In the next operation, we take the next digit, i.e., 0 and divide 2560 by $3(10a+b)^2=588$. The quotient thus obtained is c which is equalto 4 here and the remainder comes out to be 208. Carrying the next 0 we have 2080. Now $3(10a+b)c^2$, i.e., 672 is subtracted from this number which gives us 1408. We take one more 0, 1408 comes 14080. Subtract $c^3(=64)$ from this number we obtained 14016. Again, we take 0 as next digit and makes 14016 as 140160. Now divide this number by $3(100a+10b+c)^2=62208$. The quotient thus obtained is d which is equal to 2 and the remainder comes 15744. We can further append one 0 to this number and repeat the procedure. This procedure as is evident, will go indefinitely. The cuberoot of the number 3000..., thus we $a.bcd$..., i.e., 1.442....

(b) CUBE-ROOT OF DECIMAL NUMBERS :

It can be noted that the technique of *Gaṇitatilaka* to find the cube root of a whole number can also be extended to find the cube

root of a decimal number. The only differences is that the first two digits after the decimal are taken to be non cubic digits, the next digit as cubic digit. Again, the next two digits as the non- cubic digits, and the next digit as cubic digit. This procedure can be repeated by appending zeroes to obtain the desired accuracy. Let it illustrate the method by taking the number of 0.8.

Example : Compute the cube root of 0.8.

Here, it can be seen that the digit at first cubic place is 0 and the number upto this cubic place is 800. We observe that the greatest possible cube less than 800 is 729. So we claim that $a^3=729$, i.e., $a=9$. Subtracting 729 from 800 we get 71. Now we take the remaining digits of the number one by one from left to right. Thus putting one digit, namely, zero after 71, we make it 710. Next, we divide 710 by $3a^2$, i.e., 243. The quotient obtained by this division is $b(=2)$ & the remainder is 224. We take the next digit from the number & make 224 as 2240. From this 2240 subtract $3ab^2$, i.e., 108 and obtain 2132. We again take the next digit from the number & make 2132 as 21320. Now, we subtract $b^3(=8)$ from 21320 and obtain 21312. In the next operation, we take the next digit from the number, i.e., 0 & divide 213120 by $3(10a+b)^2=25392$. The quotient thus obtained is c which is equal to 8 here and the remainder comes out to be 9984. Carrying the next digit from the number we have 99840. Now $3(10a+b)c^2$, i.e., 17664 is subtracted from this number which gives us 82176. Again, taking the next digit from the number 82176 becomes 821760. Next, we subtract $c^3(=512)$ from the number & the remainder comes out to be 821248. This procedure, again will go indefinitely. As such by this procedure, the cube root of 0.8 will be 0.928..., We can terminate the procedure upto a desired accuracy. It can also be noted that the cube root of 0.8 using calcular comes out to be 0.928317766.... Using this techniques we can also solve the equation $x^2=a$.

From the above, we conclude that the above mentioned techniques for calculating cube and cube root of numbers, etc. are original and accurate. The basic difference between the methods being used these days and the methods given by in *Srīpati* in *Gaṇitatilaka* is that *Srīpati's* method are based purely on Arithmatical calculations whereas modern techniques are based on Algebrical methods.

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REFERENCES

1. A.K. Bag, Mathematics in Ancient and Medieval India, Chaukamba Orientalia, Varaasi, 1979.
2. B. Datta & A.N. Singh, History of Hindu Mathematics, (part 1 & 11), Asia Publishing House, Bombay, 1962.
3. S. Dvivedi (Ed.), *TriŚatikā* of *Śridhara*, Benares, 1899.
4. S. Dvivedi (Ed.), *Brāhmasphutasidhānta* of *Brahmagupta,* Benares, 1902.
5. R.C. Gupta, Six type of *Vedic* Mathematics, Gaṇita-*Bhāratī*, Vol. 16. 1-4, (1994), 5-15.
6. P.D. Jyautishacharya (Ed.), The Gaṇita-*Kaumudī*, Part ((&11), Benares, 1936.
7. H.R. Kapaida (Ed.), *Gaṇitatilaka of* Śrīpati, Gaekwad's Oriental Series, Baroda, 1937.
8. R.S. Lal, Cubing in ancient Hindu Mathematics, M.E. (Siwan), 18 (1984).
9. R.S. Lal, Extracting of cube root in Hindu Mathematics, M.E. (Siwan).18 (1984)

10. R.C. Pandey, *Līlavatī of Bhāskarācārya*, Chaukamba Oriental,Varanasi, 1993.

11. M. Ranga Charya (Ed.), *Gaṇita-Sāra-Sagraha* by *Mahāvira*, Madras, 1993.

12. K.N. Sinha, *Śrīpati Gaṇitatilakas*: English, Translation, *Gaṇita-Bhāratī*, 1982, 112-133.

13. S.L. Singh and R. Chand, An extension of Indian cuberoot, J. Nature Phys. Sc. Vol. 3 (1989), 78-79.

14. K.S. Shukla, *Āryabhaṭiya of Āryabhata*, Indian National Science Academy, 1976.

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट, खण्ड १ (१९९८), अंक २, २२३-२३२
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DECIDING LINK PRIORITIES IN THE DESIGN OF A COMPUTER COMMUNICATION NETWORK FOR TERMINAL RELIABILITY OPTIMIZATION

VINOD KUMAR*, K. K. AGGARWAL** and M.S. ASWAL***

Building reliability into the design of a Computer Communication Network (CCN) is the need of the hour which depends on the reliabilities of the network elements and its topological design. The topological design is basically concerned with location of computers and communication links connecting them so as to satisfy the performance requirements withthe least cost. The problem of preparing a topological layout of a CCN with optimum ST-reliability if faced by the network designer whenever a new network is set up or an old network is expanded. The present paper attempts to address the above problem by suggesting a heuristic approach for the same. The goal is achieved by deciding the physical location of computers and communication links in such a way that the network required for desired connectivity is optimized. Thus the ST-reliability of a CCN with a fixed topology is optimized for a given set of link-reliabilities. The results reported in this paper have been compared with those derived from exhaustive enumeration technique. it is observed that these results are very close to the true optimum results in all the cases.

INTRODUCTION

Computers have become essential in the day-to-day problems of most public and private enterprises. Apart from the scientific or

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business data management tasks handled within a computing environment a tremendous volume of data must be exchanged among different facilities, whether or not, they are located remotely. The need for resource sharing of specialized hardware, software data banks etc. has been long recognized. As a result, several networks have come about to satisfy the ever growing requirements in data communication and resource sharing [1].

Several issues arise with regard to topological design and operational procedure for a Computer Communication network. The reliability of the network is one of the most important criteria for performance analysis of a CCN.

The ST-reliability of a CCN heavily depends upon the reliability of communication paths between each node pair [2,3] as well as the topological layout of various computer systems, links and communication facilities. Various methods have been reported to evaluate the terminal reliability of a CCN [4-9]. However, nothing significant has been reported for optimizing networks except [3,4,10,15]. In this paper we present a heuristic approach for optimizing link reliability allocation in a fixed topology of a CCN so that, for the given source and terminal nodes, the resulting ST-reliability is near optimum. The proposed technique uses minimal path finding process given in [16]. The numerical value of ST-reliability is obtained by employing [9]. The True optimum ST-reliability values have also have been determined by exhaustive enumeration. It is observed that the optimum ST-reliabilityy values achieved through our method are very close to the true optimum values.

NOTATIONS

LS : Set of communication links

Fi : frequency of ith link

LS_i : subsets of LS

ALS : arranged link set

C_i : path cardinalities

RAV : Reliability Assignment Vector

SLR : Set of link reliabilities.

RS_T : ST-reliability as obtained on the basis of the given set of link reliabilities

OR_{ST} : Optimal ST-reliability calculated by employing the present technique

OR_{True} : True optimum ST-reliability determined by exhaustive enumeration.

ASSUMPTION

1. All computers are perfectlyy reliable.
2. Links have two states : working (On) or failed (Off).

THE PRESENT TECHNIQUE

The present approach first enumerates all minimal *ST*-paths and then arranges these paths in increasing order of their cardinalities. In the path of least cardinality, the occurrence of each link (frequency) is counted and the link-set ***LS*** is arranged in the decreasing order of the frequency of each link.

Let L_1 links have frequency F_1, L_2 links have F_2,............., L_k links have F_k in the path of the least cardinality such that $F_1 > F_2 >.....F_k$. In link set LS the links having the same frequency can be placed in any order. Further partitioning of link set LS results into subsets $LS_1, LS_2,..........LS_k$ with the cardinalities $L_1, L_2.....L_k$ respectively, each of which contains the links having equal

frequencies. These subsets are rearranged in decreasing order of frequency of each link in the paths of higher cardinalities. The concatenation of the arranged subsets $LS_i(i = 1, ..., k)$ yields the finally arranged link set ALS. The reliability, in decreasing order, is then assigned to each link on the basis of its position in ALS. Depending upon the order of link reliabilities, given in ALS, numerical value of ST-reliability is obtained by employing [9].

The numerical value of ST-reliability is also determined by taking link-reliabilities in the order as originally given. If this is higher htan the previous value, it is considered to be the optimum value of ST-reliability. Otherwise, the former is taken as the optimum value. The algorithmic representation of the whole process described above is as follows :

ALGORITHM

Step 1. Determine all minimal ST-paths by employing [16].

Step 2. Arrange these paths in the increasing order of cardinalities, say $C_1, C_2.....C_n$.

Step 3. Organize LS in decreasing order of link frequencies in the paths of the least cardinality i.e. of C_1.

Step 4. Partition LS into subsets $LS_1, LS_2 .....LS_n$ depending on the number of different link frequencies, in the paths of cardinality C_1.

Step 5. Count the frequency of each link in the paths of cardinality C_2.

Step 6. Arrange the links in each of the $LS_i(1 = 1,2, ...,k)$ in the decreasing order of their frequencies in the paths of cardinality C_2 and if required further partion each of the LS_i according to steps 3 & 4.

Step 7. Repeat steps 4 to 6 till all the paths are taken care of.

Step 8. Concatenate all subsets of LS to form ALS preserving proper order of subsets.

Step 9. Determine the terminal reliability R_{ST} on the basis of the given set of link-reliabilities.

Step 10. Determine OR_{ST} on the basis of the heuristic link assignment implemented in step 8.

Step 11. If $OR_{ST} < R_{ST}$ then $OR_{ST} = R_{ST}$.

Step 12. Stop.

IMPLEMENTATION OF THE ALGORITHM

Consider the network given in figure 1. and a given set of link reliabilities SLR={.93.96.89.85.95.88.75.94}.

Step 1.

Taking node 1 as source node and node 3 as terminal node, the paths in terms of link indices are listed below in the increasing order of their cardinalities :

1	:	2			
2	:	1	7		
3	:	6	4	8	
4	:	1	5	8	
5	:	6	3	7	
6	:	1	3	4	8
7	:	6	3	5	8
8	:	6	4	5	7

Step 2.

The number of occurrences of each link in the paths of different cardinalities is tabulated as follows :

Cardinality\link>	1	2	3	4	5	6	7	8
1	0	1	0	0	0	0	0	0
2	1	0	0	0	0	0	1	0
3	1	0	1	1	1	2	1	2
4	1	0	2	2	2	2	1	2

Step 3. The arranged link set is given as

$$LS = \{2,1,3,4,5,6,7,8\}$$

It may be observed that the frequency of link 2 is 1, and rest of the links are not present in the path of order 1. Therefore, LS may be divided into two subsets LS_1, and LS_2, having links of equal frequency in the paths of cardinality 1.

$$LS_1 = \{2\}$$

$$LS_2 = \{1,3,4,5,6,7,8\}$$

Step 4 to 6

The link set LS_2 is further rearranged and parttioned according to step 3 in decreasing order of frequency of each link in the paths of cardinality 2 which produces

$$LS_{21} = \{1,7\}$$

$$LS_{22} = \{3,4,5,6,8\}$$

Similarly, considering the paths of cardinality 3, we get subsequent partitions as :

$$LS_{21} = \{1,7\}$$

$$LS_{221} = \{6,8\} \text{ and}$$

$$LS_{222} = \{3,4,5\}$$

The link positions in LS_2 are not changed as the frequencies of link 1 and 7 are same in the paths of cardinality 3. At last, the paths of cardinality 4 are considered which do not alter the order of any of the above link sets.

Step 7 : Now, finally arranged link set, ALS, is given by

$$ALS = \{LS_1, LS_{21}, LS_{222}\} = \{2,1,7,6,8,3,4,5\}$$

Step 8: The corresponding reliability assignment vector is

$$RAV = \{.94,.96,.85,.88,.75,.93,.95,.89\}$$

Step 9: $R_{ST} = 0.9989541.$

Step 10: $OR_{ST} = 0.9994758.$

Step 11: here $R_{ST} < OR_{ST}$, therefore, final OR_{ST} remains unaltered.

CONCLUSION

A heuristic method is suggested in this paper to optimize *ST*-reliability in a computer communication network. it can be used by a network designer as an on-paper predesign exercise for preparing optimum layout of a CCN. Also, it would be helpful for the maintenance engineer to deciding as to which of the links is to be given the highest priority for maintenance purpose.

The algorithm introduced in this paper is coded into FORTRAN-77 and tested on several network topologies. The network topologies, given in figures 1,2 & 3, are considered to illustrate the result. For these CCNs, the three types of ST-reliability values are obtained and shown in tables 1,2 & 3 respectively.

One can observe that in all the cases, OR_{ST} values are very close to OR_{True} values. Also it is apparent from Table-2 that there is significant difference between R_{ST} and OR_{ST} values which indicates that a better network layout resulting into remarkably improved ST-reliability can be present heuristic technique provides tremendous saving int he computational time as compared to the exhaustive enumeration.

Table 1.

$Link \geq$	1	2	3	4	5	6	7	8	ST – Relibality
R_{ST}	0.93	0.96	0.89	0.85	0.95	0.88	0.75	0.94	0.9989541
OR_{ST}	0.95	0.96	0.88	0.85	0.75	0.93	0.94	0.89	0.9994758
OR_{TRUE}	0.94	0.96	0.85	0.88	0.75	0.93	0.95	0.89	0.9995087

Table 2.

$Link \geq$	1	2	3	4	5	6	7	8	9	ST – Relibality
R_{ST}	0.75	0.85	0.75	0.69	0.62	0.78	0.72	0.95	0.98	0.9788156
OR_{ST}	0.75	0.72	0.78	0.95	0.98	0.75	0.85	0.69	0.62	0.9996562
OR_{TRUE}	0.75	0.72	0.85	0.95	0.98	0.75	0.78	0.69	0.62	0.9997337

Table 3.

$Link \geq$	1	2	3	4	5	6	7	8	9	ST – Relibality
R_{ST}	0.72	0.69	0.75	0.98	0.88	0.79	0.95	0.96	0.98	0.9996176
OR_{ST}	0.95	0.88	0.98	0.98	0.79	0.72	0.96	0.75	0.69	0.9999596
OR_{TRUE}	0.72	0.79	0.98	0.95	0.96	0.69	0.88	0.98	0.75	0.9999657

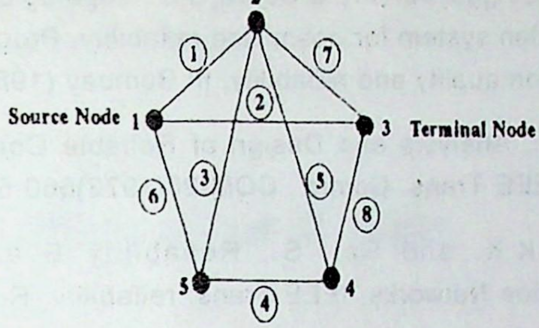

Fig. - 1

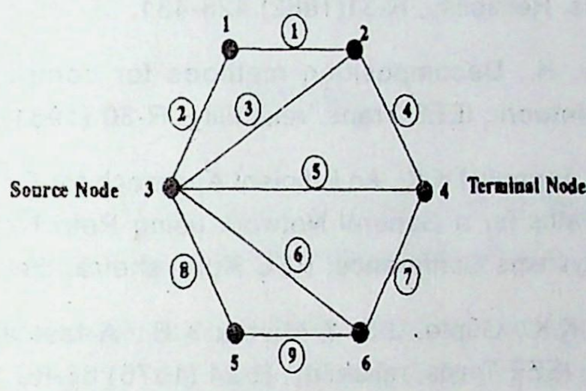

Fig. - 2

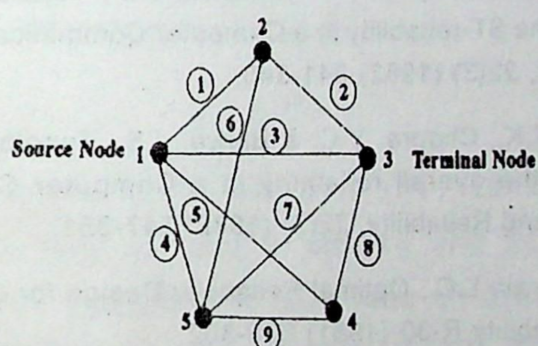

Fig. - 3

REFERENCES

1. Soi, inder M., Aggarwal, K.K., On shortest-path algorithm in the topological layout of Computer network: a comparative study, Int J. Systems Science, 12(11), (1981) 1379-1387.

2. Aggarwal, K. K., Chopra, Y.C., Bajwa, J.S., Topological layout of links for optimizing the overall reliability in Computer Communication system, Microelotron Reliab., 22(3) (1982) 347-51.

3. Chopra, Y.C., Aggarwal K.K. & Bajwa, J.S., Topological design of a Computer Communication system for optimizing reliability, Proceedings of the National Conference on quality and reliability, III Bombay (1982).

4. Wilkov, R.S., Analysis and Design of Reliable Computer Communication Networks, IEEE Trans. Comm., COM-20(1972)660-678.

5. Aggarwal, K.K. and Rai, S., Reliability Evaluation in Computer Communication Networks, IEEE Trans. reliability, R-30 (1981)32-35.

6. Rai, S., Cutset approach to Reliability Evaluation in Computer Networks, IEEE Trans. Reliability, R-31(1982) 428-431.

7. Nakazawa, H., Decomposition methods for computing the reliability of Complex Network, IEEE Trans. reliability, R-30 (1981) 289-292.

8. Kumar, V., Aggarwal K.K., An Efficient Approach for Enumerating All Systems Success Paths for a General Network using Petri Nets Proceedings of the National Systems Conference, REC Kurukshetra, Dec. 20-24 (1987) 79-82.

9. Aggarwal, K.K., Gupta, J.S. & Mishra K.B., A fast algorithm for reliability evaluation, IEEE Trans. reliability, R-24 (1975) 83-85.

10. Aggarwal, K.K., Chopra, Y.C. & Baiwa, J.S., Topological layout of links for optimizing the ST-reliability in a Computer Comminication Network, Microlect. & Reliability, 22(3) (1982) 341-345.

11. Aggarwal, K.K., Chopra, Y.C. & Bajwa, J.S., Topological Layout of links for optimizing the overall reliability in a Computer Communication system, Microelect and Reliability, 22(3) (1982) 347-351.

12. Kim, J.K., Frair, L.C., Optimal Reliability Design for Complex Systems, IEEE Trans. Reliability R-30 (1981) 300-302.

13. Weikang Tan, Qipping Zhao & Jingye He, A quantitative theory on allocation of system reliability, Acta Automatics Sincica, 9(1983) 99-105.

14. Kiu. S.W., McAlister, D.F., Reliability Optimization of Computer Communication Networks, IEEE Trans. On Reliability, 37(5) (1988) 475-483.

15. Kumar, V., Aggarwal K.K., Optimization of Terminal and Multiterminal reliabilities of Computer Communication Network, Proceedings of the National System Conference NSC-91, Roorkee, March 13-15 (1992) 197-200.

16. Jasman, G.B., Kai, O.S., New Techniques in Minimal Path and Cutest Evaluation, IEEE Trans. Reliability. 34(2) (1985) 136-143.

गुरुकुल कांगड़ी विज्ञान पत्रिका आर्यभट, खण्ड १ (१९९८), अंक २, २३३-२३४
Gurukula Kangri Vijñāna Patrikā Āryabhaṭa, Vol. 1 (1998), NO. 2, 233-234

उष्मागतिकी का दैनिक जीवन में महत्व

रजनीश दत्त कौशिक*

सारांश

उष्मागतिकी विज्ञान की वह शाखा है जिसमें उर्जा व कार्य में सम्बन्धों व एक दूसरे में परिवर्तन पर प्रकाश डाला जाता है। उष्मागतिकी के तीन नियम हैं।

इसके प्रथम नियमानुसार उर्जा के विभिन्न रूपों को एक दूसरे में बदला जा सकता है यथा विकिरण, विद्युत, कार्य, उष्मा आदि। साथ ही ऐसी कोई भी मशीन बनाना असम्भव है जो ली गई उर्जा से अधिक मात्रा में उर्जा उत्पन्न कर सके। इसी नियम को किसी संवृत निकाय हेतु निम्नवत प्रदर्शित किया जा सकता है।

$$q = \Delta E + w$$

अर्थात् किसी निकाय को दी गयी उष्मा (q), उसकी आन्तरिक उर्जा वृद्धि (ΔE) तथा उसके द्वारा किये गये कार्य में परिवर्तित होती है। आन्तरिक उर्जा किसी निकाय में संचित उर्जा है। उष्मागतिकी के प्रथम नियम को दैनिक जीवन में लगाया जा सकता है। जैसे नवजात शिशु की वृद्धि, युवावस्था में उर्जा सन्तुलन, डायबिटीज जैसी बीमारियां आदि।

नवजात शिशु को दूध, भोजन आदि के रूप में उर्जा प्राप्त होती है। इस निकाय हेतु यह "q" है। उसके द्वारा किया गया कार्य (w) लगभग शून्य है। अतः उर्जा की अधिकांश मात्रा, आन्तरिक उर्जा वृद्धि में काम आती है तथा उसके अंग-प्रत्यंगों का विकास तेजी से होता है।

किसी युवक द्वारा भोजन के रूप में ली गयी उर्जा का कुछ भाग आन्तरिक उर्जा में बदलता है व शेष मुक्त उर्जा कार्य करने में प्रयुक्त होती है। यदि कार्य करते समय

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मुक्त उर्जा के अतिरिक्त आन्तरिक उर्जा को भी कार्य में परिवर्तित किया जाता है तब युवक थकान महसूस करता है व उसकी अवस्था प्रभावित होती है। तत्पश्चात जब वह पुन: भोजन ग्रहण करता है तब उसे मिली उर्जा में से, सर्वप्रथम आन्तरिक उर्जा में आयी कमी को, पूरा किया जाता है तथा शेष उर्जा (अर्थात मुक्त उर्जा) द्वारा वह फिर से कार्य करने को तत्पर होता है।

यदि वह युवक, युवावस्था में ही, कार्य करना कम कर दे तथा भोजनादि के रूप में उर्जा ग्रहण करता रहे तो ग्रहण की गई उर्जा की अधिक मात्रा, उष्मागतिकी के प्रथम नियमानुसार, आन्तरिक उर्जा में परिवर्तित होती जायगी। इसी प्रकार की स्थिति वृद्धावस्था में भी होती है। कुछ तो इस अवस्था में शरीर के अंग कार्य कम करने लगते हैं और कुछ अनिच्छा के कारण भी, ग्रहण की गई उर्जा का अधिकांश भाग आन्तरिक उर्जा में परिवर्तित होता जाता है।

यहां यह बात ध्यान देने की है कि हमारे जीवित रहने के लिए उर्जा आवश्यक है परन्तु उर्जा की अधिक मात्रा सदैव हानिकारक है - क्या आप एटम बम द्वारा दी गयी उर्जा पसन्द करेंगे ?

अत: निकाय द्वारा कार्य कम करने पर व आन्तरिक उर्जा में अत्यधिक वृद्धि हो जाने पर हमारे शरीर पर कुप्रभाव पड़ता है तथा शरीर की विभिन्न क्रियाएं प्रभावित होती हैं व जन्म लेती हैं उर्जा से सम्बन्धित बीमारियां यथा डायबिटीज (मधुमेह)।

मधुमेह उर्जा की बीमारी है - यह बात कोई चिकित्सक अथवा जैव वैज्ञानिक स्वीकार नहीं करता उसके अनुसार यह इन्सुलिन के कारण हुई बीमारी है। परन्तु, यदि हम किसी चिकित्सक द्वारा मधुमेह पीड़ित व्यक्ति को दिये जा रहे उपचार में व्यक्त किये गये पथ्य (परहेज) तथा अन्य दी गयी सलाहों पर ध्यान दें तो पायेंगे कि यह सब उष्मागतिकी के प्रथम नियमानुसार किया जा रहा है। चिकित्सक कहता है कि भोजन में कमी करो (अर्थात् "q" कम करो), कुछ भौतिक कार्य करो यथा घूमने जाओ, व्यायाम करो (अर्थात "w" बढ़ा दो)। सप्ष्ट है कि वह रोगी को "ΔE" कम करने के लिए कह रहा है।

बालश्रम (***Child labour***) की समस्या भी इसी प्रकार स्पष्ट की जा सकती है। वह बच्चा, जो अभी बढ़ रहा है तथा उसके शारीरिक व बौद्धिक विकास हेतु आन्तरिक उर्जा की अधिक आवश्यकता है, यदि भौतिक कार्य अधिक करने लगे तब अधिकांश "q" का "w" में परिवर्तन होने के कारण "ΔE" में वांछित वृद्धि नहीं हो पायगी। ऐसे में उसके द्वारा किये गये भौतिक कार्य के कारण, हो सकता है कि वह शारीरिक रूप से हृष्ट-पुष्ट दिखाई दे, परन्तु उसका अन्यथा विकास जैसे मानसिक व बौद्धिक विकास नहीं हो पायगा। अत: बालश्रम किसी राष्ट्र के सर्वांगीण विकास हेतु उचित नहीं है।

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